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Division of Electromagnetic Research

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Diffraction of a Skew Plane Electromagnetic Wave by an Absorbing Right-Angled Wedge

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Abstract

An exact solution of Maxwell's Equations is obtained for a plane wave of arbitrary incidence striking a partially absorbing three-dimensional right-angled wedge. In addition to the exact solution, an asymptotic representation for the far field is given.

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1. Introduction

In this paper we present an exact solution for the diffraction of a skew incident electromagnetic plane wave by a three-dimensional right-angled wedge. An impedance boundary condition is prescribed on one surface of the wedge and the usual boundary condition for perfect reflection is prescribed on the other. Mathematically, this problem is of interest because, due to the mixed boundary conditions, it cannot be treated by the standard method of separation of variables. Physically, the problem is of importance since it gives the solution for a right-angled wedge having one face that acts like a partially-absorbing surface, i.e., a surface which absorbs a maximum amount of radiation for certain angles of incidence of the incoming plane wave. For other angles of incidence the absorption varies from the special case of maximum absorption to the case of zero absorption or perfect reflection. By suitably choosing a parameter characterizing the material of the screen, it is possible to represent, phenomenologically, a screen that absorbs normally incident radiation at a certain frequency. Hence our solution includes the important special case of a wedge with one of its faces acting like a black screen for some direction of incidence, and partially-absorbing for all other angles of incidence. The type of surface is described in more detail in the Appendix. The diffraction problem solved here is less ambitious than that desired by people who are interested in diffraction by black screens. (See Baker and Copson^[1].) On the other hand, it has the advantage of being expressible as a well-defined boundary value problem that can be solved. The method used is also applicable to the problem of diffraction by an imperfectly conducting wedge and to the study of surface wave phenomena on specially surfaced wedges. We do not, however, treat these problems here.

In the present paragraph we wish to explain the mathematical peculiarities introduced by the impedance boundary condition and to describe the method we employ for the solution of the problem. The impedance boundary condition prescribes that

certain linear combinations, $L_{1,2}$, say, of the fields and their cartesian derivatives vanish on the absorbing face of the wedge. The unknown electric fields themselves vanish on the other (perfectly conducting) face of the wedge. The angular space bounded by these faces is $\frac{3\pi}{2}$. The method we use starts with the definition of the linear combinations $L_{1,2}$ as auxiliary wave functions (or, as we later call them, transformed quantities) defined everywhere in the angular region $\frac{3\pi}{2}$. Of course these new functions obey a simple boundary condition on the absorbing face (they vanish there). But, fortunately, they also obey simple boundary conditions on the conducting face, just as did the original unknown fields. Wave functions so characterized are easy to obtain. Once the new unknown functions are obtained, we can determine the original field by quadratures. In using this method we follow an idea of Stoker [8] and Lewy [3] which was further investigated by Weinstein [10]*. They determined simplifying linear combinations L for boundary value problems in certain angular domains. These angular domains treated by Stoker and Lewy are all less than π . Greater angles seem to have no interpretation in water wave theory which occupied these authors at the time. But actually we notice that if this device simplifies boundary conditions for a domain β , it also holds for domains $\pi + \beta$, $\pi - \beta$, and $2\pi - \beta$. This fact is exploited in a paper by one of the present authors [2] who treats a scalar, two-dimensional version of our problem in the angular space $\frac{3\pi}{2} = 2\pi - \frac{\pi}{2}$ rather than Stoker's $\frac{\pi}{2}$. The boundary conditions (namely, $u_x - \lambda u = 0$ when $x = 0$, $y < 0$; $u_y = 0$ when $y = 0$, $x > 0$) and the simplifying device are essentially the same as those employed by Stoker. Necessarily the development, after simplifying the boundary conditions, differs from that occurring in water wave theory, since the type of problem to be solved is different in respect to edge conditions and conditions at infinity. The same is true of our present work which is, however,

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See reference [8] for a fuller discussion of the procedure referred to here.

considerably more difficult due to the vectorial character of the boundary conditions. In Section 3 the procedure in our vectorial problem is described more fully, with an explanation of how the method of separation of variables plus proper consideration of edge behavior leads to a unique solution under conditions which are physically meaningful.

We close this introduction by outlining the contents of the various sections. Section 2 contains a precise statement of the problem solved in this paper. In Section 3, as we have already noted, we give a summary of the method used. Section 4 contains a discussion of the transformed boundary conditions, wave equations and incident plane waves. In Section 5 we solve certain first order partial differential equations in order to express the actual field quantities in terms of the transformed field quantities. Section 6 contains certain, everywhere-finite, solutions of the transformed problems. In Section 7 we give the exact solution of the actual problem, and in Section 8 we find an asymptotic solution for the far field.

2. Statement of problem

Consider a three-dimensional right-angled wedge whose edge lies along the z -axis as shown in Figure 1. Incident on the wedge, and propagating in an arbitrary direction, is a simple harmonic plane wave of the form

$$(2.1) \quad \vec{E}^{inc} = \vec{E}^0 e^{i\xi x + i\eta y + i\zeta z - i\omega t}$$

where

$$(2.2) \quad \vec{E}^0 = \begin{pmatrix} E_x^{inc} \\ E_y^{inc} \\ E_z^{inc} \end{pmatrix}$$

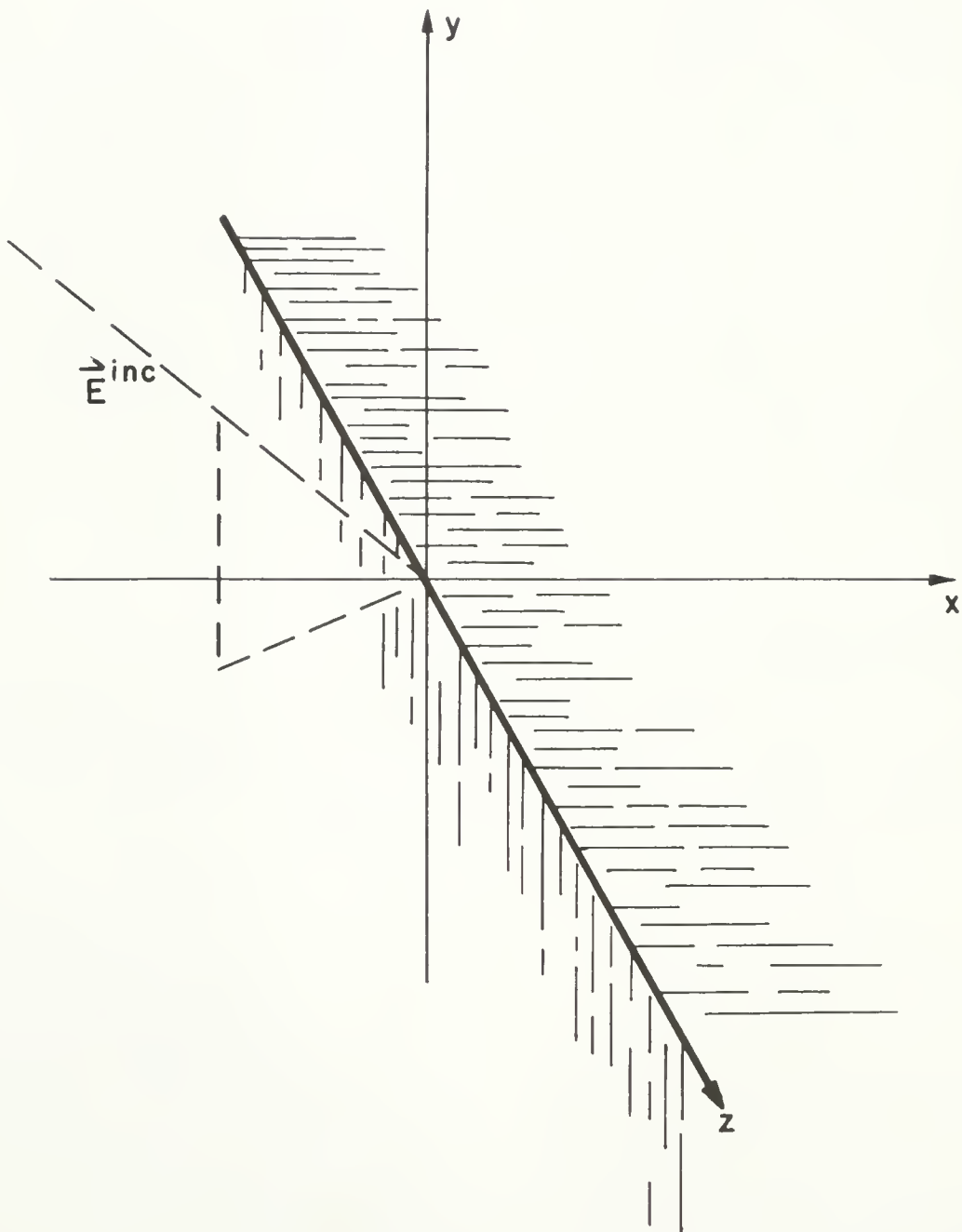


Figure 1

and

$$\begin{aligned}
 \xi &= -k \sin \alpha \cos \theta_0 \\
 \eta &= -k \sin \alpha \sin \theta_0 \\
 \zeta &= k \cos \alpha .
 \end{aligned}
 \tag{2.3}$$

In the above expressions k is the propagation constant of free space; ω is the angular frequency; t is the time; x, y, z are rectangular cartesian coordinates, and $\vec{E}$ is the electric vector. The angles α and θ_0 indicate the direction of the normal $\vec{n}$ of the incident plane wave front. (See Figure 2.) The surface of the wedge given by $x = 0, y \leq 0$ is assumed to be perfectly-conducting. The boundary condition expressing this fact is well known to be

$$\begin{pmatrix} E_x \\ E_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad y = 0, x \geq 0.
 \tag{2.4}$$

The surface given by $x = 0, y \leq 0$ is assumed to be partially-absorbing. This property we express mathematically by the boundary condition

$$\begin{pmatrix} E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 0 & Z \\ -Z & 0 \end{pmatrix} \begin{pmatrix} H_y \\ H_z \end{pmatrix} \quad x = 0, y \leq 0,
 \tag{2.5}$$

where H_y and H_z are components of the magnetic field vector. The value of Z is given by

$$Z = \frac{\omega \mu}{k} s
 \tag{2.6}$$

where μ is the permeability of free space, k is the free space propagation constant and s is a real dimensionless constant ($0 \leq s < \infty$). Note that when $s = 0$ the second boundary condition corresponds to the case of a perfect conductor since the electric field vectors E_y and E_z are zero on the surface $x = 0, y \leq 0$. The notion of a partially-absorbing surface is discussed in the Appendix where we show

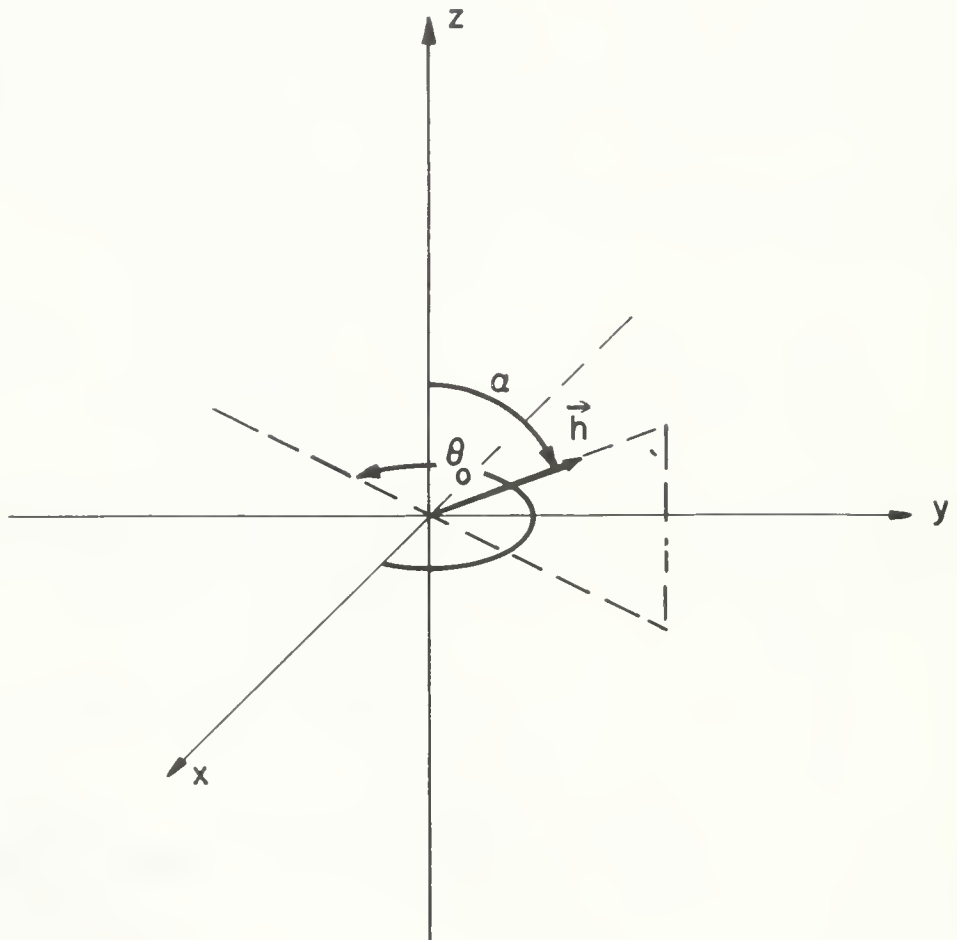


Figure 2

that the assumed form of the boundary condition is appropriate for describing this property of a surface mathematically.

The mathematical problem solved in this paper can be stated as follows: Find an exact solution of Maxwell's equations for a plane wave of arbitrary incidence striking a three-dimensional right-angled wedge subject to the boundary conditions given by (2.4) and (2.5). Maxwell's equations in free space reduce to

$$\begin{aligned} \text{curl } \vec{E} &= ik \vec{H} \\ \text{curl } \vec{H} &= -ik \vec{E} \end{aligned} \quad (2.7)$$

where the time dependence is $e^{-i\omega t}$. In addition to the exact solution, an asymptotic representation for the far field is given. In the discussion that follows it is assumed that the propagation constant k has a small positive imaginary part (that is, the conductivity of free space is not zero). This is done in order to ensure the convergence of certain integrals occurring in the analysis. Note that the electromagnetic fields in the problem are of the form

$$\begin{aligned} E_i &= \hat{E}_i(x,y) e^{i\zeta z - i\omega t} \\ H_i &= \hat{H}_i(x,y) e^{i\zeta z - i\omega t} \end{aligned} \quad (2.8)$$

since the problem is essentially that of the propagation of an electromagnetic wave along a cylindrical structure. Thus the coordinate z may be separated out of the wave equation, leaving a new propagation constant depending on both k and ζ .

3. Summary of method

The method used in solving the proposed mixed boundary value problem will now be summarized. The boundary conditions for the problem can be expressed in the following equivalent form

$$(3.1) \quad \begin{aligned} \frac{\partial E_y}{\partial y} &= 0 \\ E_z &= 0 \end{aligned} \quad y = 0, x \geq 0$$

$$(3.2) \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{pmatrix} - \lambda \begin{pmatrix} E_y \\ E_z \end{pmatrix} = 0 \quad x = 0, y \leq 0$$

where

$$(3.3) \quad \lambda = \frac{i\omega\mu}{Z}.$$

The first equation of (3.1) is an immediate consequence of equations (2.4) and the divergence condition for $\vec{E}$. The divergence condition for the electric field vector is given by

$$(3.4) \quad \text{div } \vec{E} = 0,$$

as follows from Maxwell's equations. The matrix equation given by (3.2) follows immediately from (2.5) by using one of Maxwell's equations.

Let us introduce the transformation equations

$$(3.5) \quad \begin{aligned} \xi_y &= -\lambda E_y + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \\ \xi_z &= -\lambda E_z - \frac{\partial E_x}{\partial z} + \frac{\partial E_z}{\partial x} \end{aligned},$$

which define new quantities ξ_y and ξ_z . This substitution results in an immediate simplification of the original problem in terms of the new (or transformed) variables ξ_y and ξ_z . It can be shown that the boundary conditions for the new quantities ξ_y and ξ_z are

$$(3.6) \quad \begin{aligned} \xi_y &= 0 \\ \xi_z &= 0 \end{aligned} \quad x = 0, y \leq 0$$

$$(3.7) \quad \begin{aligned} \frac{\partial \xi_y}{\partial y} &= 0 \\ \xi_z &= 0 \end{aligned} \quad y = 0, x \geq 0$$

and that the transformed variables ξ_y and ξ_z satisfy the wave equations

$$(3.8) \quad \begin{aligned} \nabla^2 \xi_y + k^2 \xi_y &= 0 \\ \nabla^2 \xi_z + k^2 \xi_z &= 0 \end{aligned} .$$

The form of the incident plane waves becomes

$$(3.9) \quad \begin{aligned} \xi_y^{inc} &= \Phi_0 e^{i\xi x + i\eta y + i\zeta z - i\omega t} \\ \xi_z^{inc} &= \Psi_0 e^{i\xi x + i\eta y + i\zeta z - i\omega t} \end{aligned}$$

where Φ_0 and Ψ_0 are related to the amplitude components of the vector plane wave $\vec{F}^{inc}$ in the original problem. Thus, by means of the transformation equations, it is possible to reduce the original problem to the solution of two separate problems involving ξ_y and ξ_z which can be solved by the method of separation of variables.

If the transformed boundary value problems involving ξ_y and ξ_z are solved, the results obtained can be used to find the solution of the original problem. This is possible since the transformation equations and the divergence condition for $\vec{E}$ can be solved explicitly for E_x , E_y and E_z in terms of ξ_y and ξ_z . That is, it can be shown that

$$(3.10) \quad E_x(x, y, z) = + \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} x} \int_{-\infty}^x e^{+\frac{k^2}{\lambda} \eta} \left[\frac{\partial \xi_y(\eta, y)}{\partial y} + \frac{\partial \xi_z(\eta, y)}{\partial z} \right] d\eta$$

$$(3.11) \quad E_y(x,y,z) = e^{\lambda x} \int_{-\infty}^x e^{-\lambda \xi} \left\{ \xi_y(\xi,y) + \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} \xi} \frac{\partial}{\partial y} \int_{-\infty}^{\xi} e^{\frac{k^2}{\lambda} \eta} \left[\frac{\partial \xi_y(\eta,y)}{\partial y} + \frac{\partial \xi_z(\eta,y)}{\partial z} \right] d\eta \right\} d\xi$$

$$(3.12) \quad E_z(x,y,z) = e^{\lambda x} \int_{-\infty}^x e^{-\lambda \xi} \left\{ \xi_z(\xi,y) + \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} \xi} \frac{\partial}{\partial z} \int_{-\infty}^{\xi} e^{\frac{k^2}{\lambda} \eta} \left[\frac{\partial \xi_y(\eta,y)}{\partial y} + \frac{\partial \xi_z(\eta,y)}{\partial z} \right] d\eta \right\} d\xi.$$

Now a few words about the nature of the solutions of the transformed problems ξ_y and ξ_z . Solutions for ξ_y and ξ_z can be found which satisfy all the conditions of the transformed problems and do not possess infinities at the edge of the wedge. Also, solutions can be found which satisfy all the conditions of the problem but are infinite at the edge. It is found that it is necessary to include the lowest order singular solution in each transformed function ξ_y and ξ_z in order to satisfy all of the boundary conditions in the original problem. The solutions finally obtained (for the actual $\vec{E}$ and $\vec{H}$ fields by including the lowest order singular terms) have infinities at the edge of the wedge which are of the same physically admissible type as those found in the problem with perfectly conducting boundary conditions.

4. The transformed boundary conditions, wave equations and incident plane waves

The prescribed boundary conditions on the wedge are the following:

$$(4.1) \quad \begin{pmatrix} E_x \\ F_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad y = 0, x \geq 0$$

$$(4.2) \quad \begin{pmatrix} E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 0 & Z \\ -Z & 0 \end{pmatrix} \begin{pmatrix} H_y \\ H_z \end{pmatrix} \quad x = 0, y \leq 0.$$

The first set of boundary conditions requires that the surface of the wedge given by $y = 0, x \geq 0$ act like a perfect conductor. The second set of boundary conditions imposes an impedance condition on the surface given by $x = 0, y \leq 0$. Note that the second set of conditions can be considered as describing the behavior of a partially-absorbing surface. (See Appendix.)

Now

$$(4.3) \quad \begin{pmatrix} E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 0 & Z \\ -Z & 0 \end{pmatrix} \begin{pmatrix} H_y \\ H_z \end{pmatrix}$$

or

$$(4.4) \quad \begin{pmatrix} E_y \\ E_z \end{pmatrix} = \frac{Z}{i\omega\mu} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{pmatrix}$$

where we have used one of Maxwell's equations. Rearranging terms, the boundary condition on the surface given by $x = 0, y \leq 0$ becomes

$$(4.5) \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{pmatrix} - \lambda \begin{pmatrix} E_y \\ E_z \end{pmatrix} = 0$$

where

$$(4.6) \quad \lambda = \frac{i\omega\mu}{Z}.$$

We shall now make the following transformation:

$$(4.7) \quad \begin{pmatrix} \xi_y \\ \xi_z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{pmatrix} - \lambda \begin{pmatrix} E_y \\ E_z \end{pmatrix} .$$

It will be shown that this transformation greatly simplifies the boundary conditions.

On the y-z plane ($y \leq 0$), the boundary conditions immediately simplify to $\xi_y = 0$ and $\xi_z = 0$ as a consequence of the substitution. Consider next the boundary conditions on the x-z plane ($x \geq 0$). We have

$$(4.8) \quad \xi_z = - \left(\frac{\partial F_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) - \lambda E_z .$$

Now $E_z = 0$ by hypothesis. Also, $\frac{\partial E_z}{\partial x} = 0$ since $F_z = 0$ and a partial derivative with respect to some variable other than y is taken. For a similar reason $\frac{\partial E_x}{\partial z} = 0$. Therefore $\xi_z = 0$. Also, we have

$$(4.9) \quad \xi_y = \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) - \lambda E_y .$$

Differentiating with respect to y, we obtain

$$(4.10) \quad \frac{\partial \xi_y}{\partial y} = \left[\frac{\partial}{\partial x} \left(\frac{\partial E_y}{\partial y} \right) - \frac{\partial^2 E_x}{\partial y^2} \right] - \lambda \left(\frac{\partial E_y}{\partial y} \right) .$$

Substituting for $\frac{\partial E_y}{\partial y}$ using the divergence condition, and $\frac{\partial^2 E_x}{\partial y^2}$ using the wave equation, we have

$$(4.11) \quad \frac{\partial \xi_y}{\partial y} = \left[\frac{\partial}{\partial x} \left\{ - \frac{\partial E_x}{\partial x} - \frac{\partial E_z}{\partial z} \right\} - \left\{ - \frac{\partial^2 E_x}{\partial x^2} - \frac{\partial^2 E_x}{\partial z^2} - k^2 E_x \right\} \right] - \lambda \left\{ - \frac{\partial E_x}{\partial x} - \frac{\partial E_z}{\partial z} \right\} .$$

The expression for $\frac{\partial \xi_y}{\partial y}$ is zero since $E_x = E_z = 0$ and there are no derivatives with respect to y . Therefore, $\frac{\partial \xi_y}{\partial y} = 0$. Summarizing, the transformed boundary conditions become

$$(4.12) \quad \begin{aligned} \xi_y &= 0 \\ \xi_z &= 0 \end{aligned} \quad x = 0, y \leq 0$$

and

$$(4.13) \quad \begin{aligned} \frac{\partial \xi_y}{\partial y} &= 0 \\ \xi_z &= 0 \end{aligned} \quad y = 0, x \geq 0.$$

It is easily shown that the transformed quantities ξ_y and ξ_z satisfy the wave equation. We have

$$(4.14) \quad (\nabla^2 + k^2) \xi_y = -\lambda(\nabla^2 + k^2)E_y + \frac{\partial}{\partial x} (\nabla^2 + k^2)E_y - \frac{\partial}{\partial y} (\nabla^2 + k^2)E_x$$

where we have used (4.7). But E_x and E_y both satisfy the wave equation. Therefore

$$(4.15) \quad (\nabla^2 + k^2) \xi_y = 0.$$

A similar argument shows that

$$(4.16) \quad (\nabla^2 + k^2) \xi_z = 0.$$

Consider an incident plane wave of the kind given by (2.1). We wish to determine the amplitudes of the transformed incident plane waves in terms of the original amplitudes $\vec{E}^{\text{inc}}$. Substituting (2.1) into (4.7), and collecting terms, we have

$$(4.17) \quad \xi_y = \Phi_0 e^{i\xi x + i\eta y + i\zeta z - i\omega t}$$

$$(4.18) \quad \xi_z = \Psi_0 e^{i\xi x + i\eta y + i\zeta z - i\omega t}$$

where

$$(4.19) \quad \Phi_0 = -\lambda E_y^{\text{inc}} - ik \cos \theta_0 \sin \alpha E_y^{\text{inc}} + ik \sin \theta_0 \sin \alpha E_x^{\text{inc}}$$

$$(4.20) \quad \Psi_0 = -\lambda E_z^{\text{inc}} - ik \cos \alpha E_x^{\text{inc}} - ik \cos \theta_0 \sin \alpha E_z^{\text{inc}}.$$

Now E_x^{inc} , E_y^{inc} and E_z^{inc} are not all independent but are related by the divergence condition

$$(4.21) \quad \xi E_x^{\text{inc}} + \eta E_y^{\text{inc}} + \zeta E_z^{\text{inc}} = 0$$

or

$$(4.22) \quad \sin \alpha \cos \theta_0 E_x^{\text{inc}} + \sin \alpha \sin \theta_0 E_y^{\text{inc}} - \cos \alpha E_z^{\text{inc}} = 0.$$

For a given problem two of the incident electric fields will be specified. The third electric field can be found from the divergence condition. In case the magnetic field is specified, the corresponding electric fields can be determined by Maxwell's equations and substituted into the expressions for Φ_0 and Ψ_0 .

5. Solution of the transformation equations.

We wish to solve the transformation equations

$$(5.1) \quad \begin{pmatrix} \xi_y \\ \xi_z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{pmatrix} - \lambda \begin{pmatrix} E_y \\ E_z \end{pmatrix}$$

for E_x , E_y and E_z in terms of ξ_y and ξ_z . Rearranging and taking partial derivatives, we have

$$(5.2) \quad \frac{\partial E_x}{\partial y} = -\xi_y - \lambda E_y + \frac{\partial E_y}{\partial x}$$

$$(5.3) \quad \frac{\partial E_x}{\partial z} = -\xi_z - \lambda E_z + \frac{\partial E_z}{\partial x}$$

or

$$(5.4) \quad \frac{\partial^2 E_x}{\partial y^2} = - \frac{\partial \xi_y}{\partial y} - \lambda \frac{\partial E_y}{\partial y} + \frac{\partial}{\partial x} \left(\frac{\partial E_y}{\partial y} \right)$$

$$(5.5) \quad \frac{\partial^2 E_x}{\partial z^2} = - \frac{\partial \xi_z}{\partial z} - \lambda \frac{\partial E_z}{\partial z} + \frac{\partial}{\partial x} \left(\frac{\partial E_z}{\partial z} \right) .$$

Adding the above expressions, we obtain

$$(5.6) \quad \left[\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \right] = - \left[\frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z} \right] - \lambda \left[\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right] + \frac{\partial}{\partial x} \left[\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right] .$$

Using the divergence condition and wave equation for E_x , the above expression simplifies to

$$(5.7) \quad \lambda \frac{\partial E_x}{\partial x} + k^2 E_x = \left[\frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z} \right] .$$

The solution of the above ordinary differential equation is

$$(5.8) \quad E_x(x, y, z) = \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} x} \int_{-\infty}^x e^{+\frac{k^2}{\lambda} \eta} \left[\frac{\partial \xi_y(\eta, y)}{\partial y} + \frac{\partial \xi_z(\eta, y)}{\partial z} \right] d\eta .$$

Now

$$(5.9) \quad \frac{\partial E_x}{\partial y} = + \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} x} \frac{\partial}{\partial y} \int_{-\infty}^x e^{+\frac{k^2}{\lambda} \eta} \left[\frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z} \right] d\eta$$

and

$$(5.10) \quad \frac{\partial E_x}{\partial z} = \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} x} \frac{\partial}{\partial z} \int_{-\infty}^x e^{+\frac{k^2}{\lambda} \eta} \left[\frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z} \right] d\eta .$$

Substituting into (5.1), we have

$$(5.11) \quad \frac{\partial E_y}{\partial x} - \lambda E_y = \xi_y + \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} x} \frac{\partial}{\partial y} \int_{-\infty}^x e^{+\frac{k^2}{\lambda} \eta} \left[\frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z} \right] d\eta$$

and

$$(5.12) \quad \frac{\partial E_z}{\partial x} - \lambda E_z = \xi_z + \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} x} \frac{\partial}{\partial z} \int_{-\infty}^x e^{+\frac{k^2}{\lambda} \eta} \left[\frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z} \right] d\eta .$$

The solutions of the above ordinary differential equations for E_y and E_z are

$$(5.13) \quad E_y(x, y, z) = e^{\lambda x} \int_{-\infty}^x e^{-\lambda \xi} \left\{ \xi_y(\xi, y) + \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} \xi} \frac{\partial}{\partial y} \int_{-\infty}^{\xi} e^{+\frac{k^2}{\lambda} \eta} \left[\frac{\partial \xi_y(\eta, y)}{\partial y} + \frac{\partial \xi_z(\eta, y)}{\partial z} \right] d\eta \right\} d\xi$$

$$(5.14) \quad E_z(x, y, z) = e^{\lambda x} \int_{-\infty}^x e^{-\lambda \xi} \left\{ \xi_z(\xi, y) + \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} \xi} \frac{\partial}{\partial z} \int_{-\infty}^{\xi} e^{+\frac{k^2}{\lambda} \eta} \left[\frac{\partial \xi_y(\eta, y)}{\partial y} + \frac{\partial \xi_z(\eta, y)}{\partial z} \right] d\eta \right\} d\xi .$$

The results obtained for E_y and E_z are expressed as iterated integrals. It is possible to express these results as single integrals by performing an integration by parts. The results obtained are

$$(5.15) \quad E_y(x, y, z) = e^{\lambda x} \int_{-\infty}^x e^{-\lambda \xi} \xi_y(\xi, y) d\xi + \frac{1}{\lambda^2 + k^2} e^{\lambda x} \int_{-\infty}^x e^{-\lambda \xi} \frac{\partial}{\partial y} \left[\frac{\partial \xi_y(\xi, y)}{\partial y} + \frac{\partial \xi_z(\xi, y)}{\partial z} \right] d\xi - \frac{1}{\lambda^2 + k^2} e^{-\frac{k^2}{\lambda} x} \int_{-\infty}^x e^{+\frac{k^2}{\lambda} \xi} \frac{\partial}{\partial y} \left[\frac{\partial \xi_y(\xi, y)}{\partial y} + \frac{\partial \xi_z(\xi, y)}{\partial z} \right] d\xi$$

and

$$(5.16) \quad E_z(x,y,z) = e^{\lambda x} \int_{-\infty}^x e^{-\lambda \xi} \mathcal{E}_z(\xi, y) d\xi + \frac{1}{\lambda^2 + k^2} e^{\lambda x} \int_{-\infty}^x e^{-\lambda \xi} \frac{\partial}{\partial x} \left[\frac{\partial \mathcal{E}_y(\xi, y)}{\partial y} + \frac{\partial \mathcal{E}_z(\xi, y)}{\partial z} \right] d\xi - \frac{1}{\lambda^2 + k^2} e^{-\frac{k^2}{\lambda} x} \int_{-\infty}^x e^{\frac{k^2}{\lambda} \xi} \frac{\partial}{\partial y} \left[\frac{\partial \mathcal{E}_y(\xi, y)}{\partial y} + \frac{\partial \mathcal{E}_z(\xi, y)}{\partial z} \right] d\xi.$$

It is easy to verify that the integral expressions for E_x , E_y and E_z satisfy the transformation equation defining $\mathcal{E}_y$ and $\mathcal{E}_z$.

In (5.8), (5.15) and (5.16) it is assumed that the transformed field quantities $\mathcal{E}_y$ and $\mathcal{E}_z$ are the most general solutions for the transformed problems. Hence they contain multiples of $\exp\left[ik(-x\cos\theta_0 \sin\alpha - y\sin\theta_0 \sin\alpha + z\cos\alpha)\right]$, the basic exponential occurring in all of our incident fields. Since k is complex, this exponential tends to destroy convergence at the lower limit for some choices of $-\frac{k^2}{\lambda}$ and $+\lambda$. The integrals still exist, however, provided that we interpret them in a particular sense. We claim that these integrals exist and yield the correct answer as long as we ignore all divergent contributions from the lower limits. The divergent contributions arise only from the incident field*. This procedure will now be justified by considering the electric field component E_x . The other field components E_y and E_z can be treated in a similar way. Consider the expression

$$(5.17) \quad E_x(x,y,z) = \frac{\exp\left[-\frac{k^2}{\lambda}x\right]}{\lambda} \int_{-\infty}^x e^{+\frac{k^2}{\lambda}\eta} \left[\frac{\partial \mathcal{E}_y^{(s)}}{\partial y} + \frac{\partial \mathcal{E}_z^{(s)}}{\partial z} \right] d\eta + E_x^{\text{inc}}$$

where $\mathcal{E}_y^{(s)}$ and $\mathcal{E}_z^{(s)}$ are the scattered parts of the total fields for $\mathcal{E}_y$ and

$\mathcal{E}_z$, respectively, and E_x^{inc} is the incident part of the electric field $E_x(x,y,z)$.

The integration clearly converges since $\text{Re } \lambda < 0$ and $\text{Re } \frac{k^2}{\lambda} > 0$. We can think of

*We note that the integrals exist in the usual sense provided $s > \sin \alpha$ and $1/s > \sin \alpha$. We introduce the new interpretation in order to remove this restriction.

performing the integration in the usual sense, where we evaluate the result at both the upper and lower limits. Or, if we choose, we can obtain the same result by evaluating the result at the upper limit and ignoring the lower limit, since the value at the lower limit in this case is zero. We observe next that if we write $E_x^{inc}(x,y,z)$ in the form

$$(5.18) \quad E_x^{inc}(x,y,z) = \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} x} \int_{-\infty}^x e^{\frac{k^2}{\lambda} \eta} \left[\frac{\partial \xi_y^{(inc)}}{\partial y} + \frac{\partial \xi_z^{(inc)}}{\partial z} \right] d\eta$$

and substitute for $\xi_y^{(inc)}$ and $\xi_z^{(inc)}$, using (4.17) through (4.20), then we obtain the incident field $E_x^{inc}(x,y,z)$ and some unwanted contribution from the lower limit. If we ignore the contribution from the lower limit we obtain the desired incident field. Hence, if we replace E_x^{inc} in (5.17) with the result given by (5.18) and ignore contributions from the lower limits in both integrals, then we can combine the two sums to give (5.8). In the special cases when $\frac{k^2}{\lambda} - ik \cos \theta_0 \sin \alpha$ or $-\lambda - ik \cos \theta_0 \sin \alpha$ are zero, we interpret the integrals as the limits resulting when $\frac{k^2}{\lambda} - ik \cos \theta_0 \sin \alpha$ or $-\lambda - ik \cos \theta_0 \sin \alpha$ approach zero. Otherwise, the integral expressions for the incident electric fields are not valid. For example, (5.18) is to be interpreted as

$$(5.19) \quad E_x^{inc}(x,y,z) = \lim_{\frac{k^2}{\lambda} - ik \cos \theta_0 \sin \alpha \rightarrow 0} \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} x} \int_{-\infty}^x e^{\frac{k^2}{\lambda} \eta} \left[\frac{\partial \xi_y^{(inc)}}{\partial y} + \frac{\partial \xi_z^{(inc)}}{\partial z} \right] d\eta$$

and (5.17) is to be interpreted as

$$(5.20) \quad E_x(x,y,z) = \lim_{\frac{k^2}{\lambda} - ik \cos \theta_0 \sin \alpha \rightarrow 0} \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} x} \int_{-\infty}^x e^{\frac{k^2}{\lambda} \eta} \left[\frac{\partial \xi_y^{(s)}}{\partial y} + \frac{\partial \xi_z^{(s)}}{\partial z} \right] d\eta + E_x^{inc}.$$

Combining these two results, we have

$$(5.21) \quad E_x(x,y,z) = \lim_{\frac{k^2}{\lambda} - ik \cos \theta_0 \sin \alpha \rightarrow 0} \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} x} \int_{-\infty}^x e^{+\frac{k^2}{\lambda} \eta} \left[\frac{\partial \xi_y}{\partial y} + \frac{\partial \xi_z}{\partial z} \right] d\eta .$$

Similar equations for $E_y(x,y,z)$ and $E_z(x,y,z)$ can be written. The reason we use total fields for ξ_y and ξ_z is that we have convenient Bessel function series which can be integrated explicitly termwise and summed in closed form.

The integral expressions that have been derived for E_x , E_y and E_z in terms of ξ_y and ξ_z must be shown to possess the properties required for the electromagnetic fields in the original problem. It is, therefore, necessary to prove that the integral expressions for E_i satisfy the wave equation

$$(5.22) \quad \nabla^2 E_i + k^2 E_i = 0 \quad (i = x, y, z)$$

and also the divergence condition

$$(5.23) \quad \frac{\partial F_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 .$$

That E_i satisfies the wave equation is easily demonstrated by substituting the integral expressions for E_i into the wave equation and integrating by parts. The proof that E_i satisfies the divergence condition is a little more involved but can be carried out in a similar way.

6. Diffraction of a plane electromagnetic wave by a wedge; Transformed problems for ξ_y and ξ_z .

It has been shown, in Section 4, that ξ_y satisfies the wave equation

$$(6.1) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \xi_y = 0$$

and the following boundary conditions

$$(6.2) \quad \frac{\partial \xi_y}{\partial y} = 0 \quad y = 0, x \geq 0$$

$$\xi_y = 0 \quad x = 0, y \leq 0.$$

Also, the incident plane wave is given by

$$(6.3) \quad \xi_y^{inc} = \bar{\Phi}_0 e^{i\xi x + i\eta y + i\zeta z - i\omega t}$$

where

$$(6.4) \quad \bar{\Phi}_0 = -\lambda E_y^{inc} - ik \cos \theta_0 \sin \alpha E_y^{inc} + ik \sin \theta_0 \sin \alpha E_x^{inc}.$$

Note that in the original problem we expect the z-variation of the electromagnetic field to be of the form $e^{i\zeta z}$, where $\zeta = k \cos \alpha$. The transformed field ξ_y must also be of the same form. Therefore,

$$(6.5) \quad \xi_y(x, y, z) = \phi(x, y) e^{i\zeta z - i\omega t}$$

and the wave equation for ϕ becomes

$$(6.6) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \kappa^2 \right) \phi = 0$$

where

$$(6.7) \quad \kappa^2 = k^2 - \zeta^2 = k^2 \sin^2 \alpha.$$

The transformed problem, therefore, reduces to a two-dimensional problem with propagation constant $k \sin \alpha$. The problem can be solved by separation of variables in polar coordinates.

Consider a two-dimensional right-angled wedge with a line source located at the point (r_0, θ_0) . We wish to solve the scalar wave equation for the case when the wave function satisfies the same boundary conditions as those given in (6.2). We wish to find, therefore, a solution of

$$(6.8) \quad \left[\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \kappa^2 \right] \phi(r, \theta; r_0, \theta_0) = - \frac{\delta(r-r_0)}{r} \delta(\theta - \theta_0)$$

that satisfies the following boundary conditions:

$$(6.9) \quad \begin{aligned} \phi_{\theta} &= 0, & \theta &= 0 \\ \phi &= 0, & \theta &= + \frac{3\pi}{2}. \end{aligned}$$

Also, we require that the scattered wave behaves like an outgoing traveling wave at infinity. When this problem is solved, the desired solution of the transformed problem for the incident plane wave given by (6.3) can be found by allowing the point source to approach infinity in the proper way. We omit the details of the analysis since the method used is the same as that given by Sollfrey^[6] for the case in which the boundary conditions are the same on each face. (Either the function or the normal derivative vanishes.) The desired solution of the transformed problem for ζ_y is given by

$$(6.10) \quad \zeta_y(x, y, z) = \phi(x, y) e^{i\kappa z - i\omega t}$$

where

$$(6.11) \quad \phi(x, y) = \frac{8}{3} \bar{Q}_0 \sum_{n=0}^{\infty} J_{\frac{2n+1}{3}}(\kappa r) \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \cos \frac{2n+1}{3} \theta,$$

and x and y are the usual polar coordinates given by $x = r \cos \theta$ and $y = r \sin \theta$.

We now need the solution of the transformed problem for ζ_z . It has been shown that ζ_z satisfies the wave equation

$$(6.12) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \kappa^2 \right) \zeta_z = 0$$

and the following boundary conditions

$$(6.13) \quad \begin{aligned} \xi_z &= 0 & y &= 0, x \geq 0 \\ \xi_z &= 0 & x &= 0, y \leq 0. \end{aligned}$$

Also, the incident plane wave is given by

$$(6.14) \quad \xi_z^{inc} = \Psi_0 e^{i\xi x + i\eta y + i\zeta z - i\omega t}$$

where

$$(6.15) \quad \Psi_0 = -\lambda E_z^{inc} - ik \cos \alpha E_x^{inc} - ik \cos \theta_0 \sin \alpha E_z^{inc}.$$

We expect the transformed field ξ_z to be of the form

$$(6.16) \quad \xi_z(x, y, z) = \Psi(x, y) e^{i\zeta z - i\omega t}.$$

The wave equation for Ψ then becomes

$$(6.17) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \kappa^2 \right) \Psi = 0$$

where κ^2 is given by (6.7). The desired solution of the transformed problem for the incident plane wave given by (6.14) can be found, using the same procedure as that used for ξ_y . We find that

$$(6.18) \quad \xi_z(x, y, z) = \Psi(x, y) e^{i\zeta z - i\omega t}$$

where

$$(6.19) \quad \Psi(x, y) = \frac{8}{3} \Psi_0 \sum_{n=1}^{\infty} J_{\frac{2n}{3}}(\kappa r) \exp \left[-i \left(\frac{2n}{3} \right) \frac{\pi}{2} \right] \sin \frac{2n}{3} \theta_0 \sin \frac{2n}{3} \theta.$$

7. Determination of the original electric fields

The electric fields for the original problem can be expressed in terms of the transformed potentials $\phi(x, y)$ and $\Psi(x, y)$ as follows:

$$(7.1) \quad E_x(x,y,z) = \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} x} e^{izk \cos \alpha} \int_{-\infty}^x e^{+\frac{k^2}{\lambda} \eta} \left[\frac{\partial \phi}{\partial y} + ik \cos \alpha \Psi \right] d\eta$$

$$(7.2) \quad E_y(x,y,z) = e^{\lambda x} e^{izk \cos \alpha} \int_{-\infty}^x e^{-\lambda \xi} \left\{ \phi + \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} \xi} \frac{\partial}{\partial y} \int_{-\infty}^{\xi} e^{+\frac{k^2}{\lambda} \eta} \left[\frac{\partial \phi}{\partial y} + ik \cos \alpha \Psi \right] d\eta \right\} d\xi$$

$$(7.3) \quad E_z(x,y,z) = e^{\lambda x} e^{izk \cos \alpha} \int_{-\infty}^x e^{-\lambda \xi} \left\{ \Psi + \frac{1}{\lambda} e^{-\frac{k^2}{\lambda} \xi} ik \cos \alpha \int_{-\infty}^{\xi} e^{+\frac{k^2}{\lambda} \eta} \left[\frac{\partial \phi}{\partial y} + ik \cos \alpha \Psi \right] d\eta \right\} d\xi .$$

In the above and all further expressions involving the electric fields, the time factor $e^{-i\omega t}$ will be omitted for convenience. Let

$$(7.4) \quad \phi(x,y) = \phi_F(x,y) + c_1 \phi_S(x,y)$$

$$(7.5) \quad \Psi(x,y) = \Psi_F(x,y) + c_2 \Psi_S(x,y).$$

In the above, ϕ_F is the solution of the transformed problem for ξ_y discussed in Section 6. It is finite at the edge of the wedge. ϕ_S is a solution of the transformed problem for ξ_y which satisfies all the conditions of the problem but has a singularity at the edge of the wedge. Ψ_F and Ψ_S are solutions for the transformed problem involving ξ_z with similar behavior. The class of singular solutions can be found from the finite solutions by replacing the Bessel functions with Hankel functions of the first kind. Among the class of transformed singular solutions only the lowest order singular solution is admissible. The lowest order singular solutions, when substituted into (7.1), (7.2) and (7.3), introduce singular terms in the original electric field $\vec{E}$, but the nature of the singularities is so mild that the energy of the electromagnetic field at the edge of the wedge

is finite. All other singular solutions are so strong that their inclusion would make the energy in the original electromagnetic field infinite at the edge of the wedge. These solutions, therefore, must be rejected on physical grounds. We find that it is necessary to use the lowest order singular solutions since they make it possible to introduce two additional constants c_1 and c_2 that can be adjusted to satisfy the boundary conditions in the original problem. It is found that only two constants are necessary, so all the admissible singular solutions are used. Hence, the solution obtained for the original problem is unique. The explicit form of the solutions is given below:

$$(7.6) \quad \phi_F(x,y) = \frac{8}{3} \Phi_0 \sum_{n=0}^{\infty} J_{\frac{2n+1}{3}}(rksina) \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \cos \frac{2n+1}{3} \theta$$

$$(7.7) \quad \phi_S(x,y) = H_{\frac{1}{3}}^{(1)}(rksina) \cos \frac{1}{3} \theta$$

$$(7.8) \quad \Psi_F(x,y) = \frac{8}{3} \Psi_0 \sum_{n=1}^{\infty} J_{\frac{2n}{3}}(rksina) \exp \left[-i \left(\frac{2n}{3} \right) \frac{\pi}{2} \right] \sin \frac{2n}{3} \theta_0 \sin \frac{2n}{3} \theta$$

$$(7.9) \quad \Psi_S(x,y) = H_{\frac{2}{3}}^{(1)}(rksina) \sin \frac{2}{3} \theta .$$

We shall now obtain an expression for the electric field E_x and impose the boundary condition $E_x(x,y,z) = 0$ when $y = 0, x \geq 0$. It is easily shown that

$$(7.10) \quad \frac{\partial}{\partial y} \left\{ H_{\frac{1}{3}}^{(1)}(rksina) \cos \frac{\theta}{3} \right\} = \frac{\partial}{\partial x} \left\{ H_{\frac{1}{3}}^{(1)}(rksina) \sin \frac{\theta}{3} \right\} + (ksina) \exp(i \frac{2}{3} \pi) \\ \cdot \left\{ H_{\frac{2}{3}}^{(1)}(rksina) \sin \frac{2}{3} \theta \right\} .$$

Therefore,

$$(7.11) \quad \int_{-\infty}^x e^{+\frac{k^2}{\lambda}\eta} \frac{\partial}{\partial y} \phi_S d\eta = \int_{-\infty}^x e^{+\frac{k^2}{\lambda}\eta} \frac{\partial}{\partial \eta} \left\{ H_{\frac{1}{3}}^{(1)}(rksina) \sin \frac{\theta}{3} \right\} d\eta \\ + (ksina) \exp(i \frac{2}{3} r) \int_{-\infty}^x e^{+\frac{k^2}{\lambda}\eta} H_{\frac{2}{3}}^{(1)}(rksina) \sin \frac{2}{3} \theta d\eta .$$

The first integral on the right-hand side of the above equation can be integrated by parts. Equation (7.11) then becomes

$$(7.12) \quad \int_{-\infty}^x e^{+\frac{k^2}{\lambda}\eta} \frac{\partial}{\partial y} \phi_S d\eta = e^{+\frac{k^2}{\lambda}x} H_{\frac{1}{3}}^{(1)}(rksina) \sin \frac{\theta}{3} - \frac{k^2}{\lambda} \int_{-\infty}^x e^{+\frac{k^2}{\lambda}\eta} H_{\frac{1}{3}}^{(1)}(rksina) \sin \frac{\theta}{3} d\eta \\ + (ksina) \exp(i \frac{2}{3} r) \int_{-\infty}^x e^{+\frac{k^2}{\lambda}\eta} H_{\frac{2}{3}}^{(1)}(rksina) \sin \frac{2\theta}{3} d\eta .$$

Using a similar procedure and the formula

$$(7.13) \quad \frac{\partial}{\partial y} \left\{ J_{\frac{2n+1}{3}}(rksina) \cos \frac{2n+1}{3} \theta \right\} = \frac{\partial}{\partial x} \left\{ J_{\frac{2n+1}{3}}(rksina) \sin \frac{2n+1}{3} \theta \right\} \\ - k \sin a \sin \frac{2}{3} (n-1)\theta J_{\frac{2}{3} (n-1)}(rksina),$$

we find that

$$\begin{aligned}
(7.14) \quad & \int_{-\infty}^x e^{\frac{k^2}{\lambda}\gamma} \frac{\partial}{\partial \gamma} \theta_F d\gamma = + \frac{8}{3} \Phi_0 e^{\frac{k^2}{\lambda}x} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 J_{2n+1} (rksinc) \sin \frac{2n+1}{3} \theta \\
& - \frac{8}{3} \Phi_0 \frac{k^2}{\lambda} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \int_{-\infty}^x e^{\frac{k^2}{\lambda}\gamma} J_{2n+1} (rksinc) \sin \frac{2n+1}{3} \theta d\gamma \\
& - \frac{8}{3} \Phi_0 k \sin a \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \int_{-\infty}^x e^{\frac{k^2}{\lambda}\gamma} J_{\frac{2}{3}(n-1)} (rksinc) \sin \frac{2}{3} (n-1) \theta d\gamma .
\end{aligned}$$

Substituting the finite and singular transformed solutions into (7.1), and making use of (7.12) and (7.14), we obtain the following expression for $E_x(x, y, z)$:

$$\begin{aligned}
(7.15) \quad E_x(x, y, z) = & \frac{8}{3} \Phi_0 \frac{1}{\lambda} e^{izk \cos a} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 J_{2n+1} (rksinc) \sin \frac{2n+1}{3} \theta \\
& - \frac{8}{3} \Phi_0 \frac{k^2}{\lambda} e^{izk \cos a} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 e^{\frac{k^2}{\lambda}x} \int_{-\infty}^x e^{\frac{k^2}{\lambda}\gamma} J_{2n+1} (rksinc) \sin \frac{2n+1}{3} \theta d\gamma \\
& - \frac{8}{3} \Phi_0 \frac{k \sin a}{\lambda} e^{izk \cos a} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 e^{\frac{k^2}{\lambda}x} \int_{-\infty}^x e^{\frac{k^2}{\lambda}\gamma} J_{\frac{2}{3}(n-1)} (rksinc) \sin \frac{2}{3} (n-1) \theta d\gamma \\
& + \frac{8}{3} \Phi_0 \frac{1}{\lambda} k \cos a e^{izk \cos a} \sum_{n=1}^{\infty} \exp \left[-i \left(\frac{2n}{3} \right) \frac{\pi}{2} \right] \sin \frac{2n}{3} \theta_0 e^{\frac{k^2}{\lambda}x} \int_{-\infty}^x e^{\frac{k^2}{\lambda}\gamma} J_{2n} (rksinc) \sin \frac{2n}{3} \theta d\gamma \\
& + c_1 \frac{1}{\lambda} e^{izk \cos a} H_1^{(1)} (rksinc) \sin \frac{\theta}{3} - c_1 \frac{k^2}{\lambda} e^{izk \cos a} e^{\frac{k^2}{\lambda}x} \int_{-\infty}^x e^{\frac{k^2}{\lambda}\gamma} H_1^{(1)} (rksinc) \sin \frac{\theta}{3} d\gamma \\
& + c_1 \frac{k \sin a}{\lambda} e^{\frac{i}{3}\pi} e^{izk \cos a} e^{\frac{k^2}{\lambda}x} \int_{-\infty}^x e^{\frac{k^2}{\lambda}\gamma} H_2^{(1)} (rksinc) \sin \frac{2\theta}{3} d\gamma \\
& + c_2 \frac{1}{\lambda} k \cos a e^{izk \cos a} e^{\frac{k^2}{\lambda}x} \int_{-\infty}^x e^{\frac{k^2}{\lambda}\gamma} H_2^{(1)} (rksinc) \sin \frac{2}{3} \theta d\gamma .
\end{aligned}$$

Let us now impose the boundary condition $E_x(x,y,z) = 0$ when $y = 0, x \geq 0$.

The surface over which the boundary condition is prescribed can be expressed in polar coordinates as $\theta = 0, r \geq 0$. Consequently, the two terms of

$E_x(x,y,z)$ not involving integrals are zero on the boundary surface since

these two terms involve the factors $\sin \frac{2n+1}{3} \theta$ and $\sin \frac{\theta}{3}$, which are both

zero when θ is zero. Consider next the terms involving integrals. We note

that all of these terms can be written as the sum of two integrals with

limits of integration $-\infty$ to 0 and 0 to $+x$, respectively. Since we have

imposed the boundary conditions on the surface $y = 0, x \geq 0$, the integration

variable η is actually the x -axis. Therefore, in polar coordinates, when

η lies between $-\infty$ and 0, $r \geq 0$ and $\theta = \pi$. When η lies between 0 and $+x$,

$r \geq 0$ and $\theta = 0$. Also, we note that $r = \sqrt{x^2} = |x|$ or $r = |\eta|$ since $y = 0$.

Therefore, we can substitute $|\eta|$ for r in all of the integrals where r

appears. It is easily seen that all integrals with limits 0 to $+x$ are zero

since $\theta = 0$. The remaining integrals with limits $-\infty$ to 0 have the terms

$e^{izk\cos\alpha}$ and $\exp(-\frac{k^2}{\lambda} x)$ as common factors. By eliminating the common factors,

the resulting expression involves constants only. This gives us one equation

between the unknowns c_1 and c_2 . We obtain

$$(7.16) \quad c_1 \left[-\frac{k^2}{\lambda^2} \int_{-\infty}^0 e^{+\frac{k^2}{\lambda}\gamma} H_1^{(1)}(|\gamma| k \sin \alpha) \sin \frac{\pi}{3} d\gamma + \frac{k \sin \alpha}{\lambda} e^{i\frac{2}{3}\pi} \int_{-\infty}^0 e^{+\frac{k^2}{\lambda}\gamma} H_2^{(1)}(|\gamma| k \sin \alpha) \sin \frac{2\pi}{3} d\gamma \right] \\ + c_2 \left[+\frac{k \cos \alpha}{\lambda} \int_{-\infty}^0 e^{+\frac{k^2}{\lambda}\gamma} H_2^{(1)}(|\gamma| k \sin \alpha) \sin \frac{2\pi}{3} d\gamma \right]$$

$$- \left[+\frac{8}{3} \Phi_0 \frac{k^2}{\lambda^2} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \int_{-\infty}^0 e^{+\frac{k^2}{\lambda}\gamma} J_{\frac{2n+1}{3}}(|\gamma| k \sin \alpha) \sin \frac{2n+1}{3} \pi d\gamma \right. \\ \left. + \frac{8}{3} \Phi_0 \frac{k \sin \alpha}{\lambda} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \int_{-\infty}^0 e^{+\frac{k^2}{\lambda}\gamma} J_{\frac{2}{3}(n-1)}(|\gamma| k \sin \alpha) \sin \frac{2}{3}(n-1) \pi d\gamma \right. \\ \left. - \frac{8}{3} \Psi_0 \frac{k \cos \alpha}{\lambda} \sum_{n=1}^{\infty} \exp \left[-i \left(\frac{2n}{3} \right) \frac{\pi}{2} \right] \sin \frac{2n}{3} \theta_0 \int_{-\infty}^0 e^{+\frac{k^2}{\lambda}\gamma} J_{\frac{2n}{3}}(|\gamma| k \sin \alpha) \sin \frac{2n}{3} \pi d\gamma \right] = 0$$

which is an expression of the form

$$(7.17) \quad c_1 A_1 + c_2 B_1 = D_1$$

where A_1 , B_1 and D_1 are known constants.

Consider next the electric field $E_z(x,y,z)$ given by (7.3). If we make use of (7.1) for E_x , the expression for $E_z(x,y,z)$ can be written

$$(7.18) \quad E_z(x,y,z) = e^{\lambda x} e^{izk \cos \alpha} \int_{-\infty}^x e^{-\lambda \xi} \Psi(\xi, y) d\xi + ik \cos \alpha e^{\lambda x} \int_{-\infty}^x e^{-\lambda \xi} E_x(\xi, y, z) d\xi$$

or

$$(7.19) \quad E_z(x,y,z) = e^{\lambda x} e^{izk \cos \alpha} \int_{-\infty}^x e^{-\lambda \xi} \Psi(\xi, y) d\xi + ik \cos \alpha e^{\lambda x} e^{izk \cos \alpha} \int_{-\infty}^x e^{-\lambda \xi} \hat{E}_x(\xi, y) d\xi$$

where

$$(7.20) \quad \hat{E}_x(\xi, y) = e^{-izk \cos \alpha} E_x(\xi, y, z).$$

Note that $\hat{E}_x(\xi, y)$ is not a function of z since all terms of $E_x(\xi, y, z)$ contain the factor $e^{+izk \cos \alpha}$. Therefore,

$$(7.21) \quad E_z(x,y,z) = e^{izk \cos \alpha} e^{\lambda x} \int_{-\infty}^0 e^{-\lambda \xi} \Psi(\xi, y) d\xi + e^{izk \cos \alpha} e^{\lambda x} \int_0^x e^{-\lambda \xi} \Psi(\xi, y) d\xi \\ + ik \cos \alpha e^{izk \cos \alpha} e^{\lambda x} \int_{-\infty}^0 e^{-\lambda \xi} \hat{E}_x(\xi, y) d\xi \\ + ik \cos \alpha e^{izk \cos \alpha} e^{\lambda x} \int_0^x e^{-\lambda \xi} \hat{E}_x(\xi, y) d\xi.$$

Let us now impose the boundary condition $E_z(x,y,z) = 0$ when $y = 0$, $x \geq 0$. In the integrals ranging from 0 to $+x$, the angle θ has the value zero. In the integrals ranging from $-\infty$ to 0, the angle θ has the value π . It is easily seen that all integrals with limits 0 to $+x$ are zero. Consequently, (7.21) reduces to

$$(7.22) \quad \int_{-\infty}^0 e^{-\lambda \xi} \Psi(\xi, 0) d\xi + ik \cos \alpha \int_{-\infty}^0 e^{-\lambda \xi} \hat{E}_x(\xi, 0) d\xi = 0.$$

The above integral expressions are constants and give us another equation between the unknowns c_1 and

c_2 . Substituting for $\mathbf{V}(\xi, 0)$ and $\hat{\mathbf{E}}_x(\xi, 0)$, setting $\theta = \pi$ and collecting terms, we obtain

$$\begin{aligned}
 (7.23) \quad c_1 & \left[-\frac{1}{\lambda} ik \cos a \sin \frac{\pi}{3} \int_{-\infty}^0 e^{-\lambda \xi} H_1^{(1)}(|\xi| k \sin a) d\xi + \frac{k^2}{\lambda^2} ik \cos a \sin \frac{\pi}{3} \int_{-\infty}^0 \exp \left[-\frac{\lambda^2 + k^2}{\lambda} \xi \right] \int_{-\infty}^{\xi} e^{+\frac{k^2}{\lambda} \eta} H_1^{(1)}(|\eta| k \sin a) d\eta d\xi \right. \\
 & \quad \left. - \frac{k \sin a}{\lambda} ik \cos a e^{\frac{1}{3} \frac{2}{3} \pi} \sin \frac{2}{3} \pi \int_{-\infty}^0 \exp \left[-\frac{\lambda^2 + k^2}{\lambda} \xi \right] \int_{-\infty}^{\xi} e^{+\frac{k^2}{\lambda} \eta} H_2^{(1)}(|\eta| k \sin a) d\eta d\xi \right] \\
 & + c_2 \left[-\sin \frac{2\pi}{3} \int_{-\infty}^0 e^{-\lambda \xi} H_2^{(1)}(|\xi| k \sin a) d\xi - \frac{(ik \cos a)^2}{\lambda} \sin \frac{2}{3} \pi \int_{-\infty}^0 \exp \left[-\frac{\lambda^2 + k^2}{\lambda} \xi \right] \int_{-\infty}^{\xi} e^{+\frac{k^2}{\lambda} \eta} H_2^{(1)}(|\eta| k \sin a) d\eta d\xi \right] \\
 & = \left[+ \frac{8}{3} \Psi_0 \sum_{n=1}^{\infty} \exp \left[-i \left(\frac{2n}{3} \right) \frac{\pi}{2} \right] \sin \frac{2n}{3} \theta_0 \sin \frac{2n}{3} \pi \int_{-\infty}^0 e^{-\lambda \xi} J_{2n}^{(1)}(|\xi| k \sin a) d\xi \right. \\
 & \quad + \frac{8}{3} \Psi_0 \frac{(ik \cos a)^2}{\lambda} \sum_{n=1}^{\infty} \exp \left[-i \left(\frac{2n}{3} \right) \frac{\pi}{2} \right] \sin \frac{2n}{3} \theta_0 \sin \frac{2n}{3} \pi \int_{-\infty}^0 \exp \left[-\frac{\lambda^2 + k^2}{\lambda} \xi \right] \int_{-\infty}^{\xi} e^{+\frac{k^2}{\lambda} \eta} J_{2n}^{(1)}(|\eta| k \sin a) d\eta d\xi \\
 & \quad + \frac{8}{3} \bar{\Psi}_0 \frac{ik \cos a}{\lambda} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \sin \frac{2n+1}{3} \pi \int_{-\infty}^0 e^{-\lambda \xi} J_{2n+1}^{(1)}(|\xi| k \sin a) d\xi \\
 & \quad - \frac{8}{3} \bar{\Psi}_0 \frac{k^2}{\lambda^2} ik \cos a \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \sin \frac{2n+1}{3} \pi \int_{-\infty}^0 \exp \left[-\frac{\lambda^2 + k^2}{\lambda} \xi \right] \int_{-\infty}^{\xi} e^{+\frac{k^2}{\lambda} \eta} J_{2n+1}^{(1)}(|\eta| k \sin a) d\eta d\xi \\
 & \quad \left. - \frac{8}{3} \bar{\Psi}_0 \sin a \frac{ik \cos a}{\lambda} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \sin \frac{2}{3} (n-1) \pi \int_{-\infty}^0 \exp \left[-\frac{\lambda^2 + k^2}{\lambda} \xi \right] \int_{-\infty}^{\xi} e^{+\frac{k^2}{\lambda} \eta} J_{2(n-1)}^{(1)}(|\eta| k \sin a) d\eta d\xi \right]
 \end{aligned}$$

which is an expression of the form

$$(7.24) \quad c_1 A_2' + c_2 B_2' = D_2'$$

where A_2' , B_2' and D_2' are known constants.

It is possible to express the constants A_2' , B_2' and D_2' in terms of single integrals instead of iterated integrals. We note that all of the iterated integrals appearing in (7.23) are of the same type. By interchanging the order of integration, it is easily shown that

$$(7.25) \quad \int_{-\infty}^0 \exp\left[-\frac{\lambda^2+k^2}{\lambda} \xi\right] \int_{-\infty}^{\xi} e^{+\frac{k^2}{\lambda}\eta} Z(\eta k \sin \alpha) d\eta d\xi = \frac{-\lambda}{\lambda^2+k^2} \int_{-\infty}^0 e^{+\frac{k^2}{\lambda}\eta} Z(\eta k \sin \alpha) d\eta \\ + \frac{\lambda}{\lambda^2+k^2} \int_{-\infty}^0 e^{-\lambda\xi} Z(\xi k \sin \alpha) d\xi,$$

where Z is either a Bessel function or a Hankel function. Substituting into (7.23), we obtain a result involving only single integrals. This result involves integrals with the exponential factors $\exp\left[+\frac{k^2}{\lambda}\eta\right]$ and $e^{-\lambda\eta}$. If the quantity $ik \cos \alpha \left(\frac{\lambda}{\lambda^2+k^2}\right)$ is factored out of the integrals involving the exponential factor $\exp\left[+\frac{k^2}{\lambda}\eta\right]$, the resulting expression is the same as that given on the left-hand side of (7.16). Hence, that part of (7.23) involving integrals with the exponential factor $\exp\left[+\frac{k^2}{\lambda}\eta\right]$ is zero. The expression given by (7.23) can therefore be simplified to

$$(7.26) \quad c_1 \left[-\frac{ik \cos a}{\lambda} \sin \frac{\pi}{3} \int_{-\infty}^0 e^{-\lambda \xi} H_1^{(1)}(|\xi| k \sin a) d\xi + \frac{ik \cos a}{\lambda} k^2 \sin \frac{\pi}{3} \left(\frac{\lambda}{\lambda^2 + k^2} \right) \int_{-\infty}^0 e^{-\lambda \eta} H_1^{(1)}(|\eta| k \sin a) d\eta \right.$$

$$\left. - \frac{ik \cos a}{\lambda} k \sin a e^{\frac{i}{2} \frac{\pi}{3}} \sin \frac{2\pi}{3} \left(\frac{\lambda}{\lambda^2 + k^2} \right) \int_{-\infty}^0 e^{-\lambda \eta} H_2^{(1)}(|\eta| k \sin a) d\eta \right]$$

$$+ c_2 \left[-\sin \frac{2\pi}{3} \int_{-\infty}^0 e^{-\lambda \xi} H_2^{(1)}(|\xi| k \sin a) d\xi - \frac{(ik \cos a)^2}{\lambda} \sin \frac{2\pi}{3} \left(\frac{\lambda}{\lambda^2 + k^2} \right) \int_{-\infty}^0 e^{-\lambda \eta} H_2^{(1)}(|\eta| k \sin a) d\eta \right]$$

$$= \left[\frac{8}{3} \Psi_0 \sum_{n=1}^{\infty} \exp \left[-i \left(\frac{2n}{3} \right) \frac{\pi}{2} \right] \sin \frac{2n}{3} \theta_0 \sin \frac{2n}{3} \pi \int_{-\infty}^0 e^{-\lambda \xi} J_{\frac{2n}{3}}(|\xi| k \sin a) d\xi \right.$$

$$+ \frac{8}{3} \Psi_0 \frac{(ik \cos a)^2}{\lambda} \sum_{n=1}^{\infty} \exp \left[-i \left(\frac{2n}{3} \right) \frac{\pi}{2} \right] \sin \frac{2n}{3} \theta_0 \sin \frac{2n}{3} \pi \left(\frac{\lambda}{\lambda^2 + k^2} \right) \int_{-\infty}^0 e^{-\lambda \eta} J_{\frac{2n}{3}}(|\eta| k \sin a) d\eta$$

$$+ \frac{8}{3} \Phi_0 \frac{ik \cos a}{\lambda} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \sin \frac{2n+1}{3} \pi \int_{-\infty}^0 e^{-\lambda \xi} J_{\frac{2n+1}{3}}(|\xi| k \sin a) d\xi$$

$$- \frac{8}{3} \Phi_0 \frac{k^2}{\lambda} \frac{ik \cos a}{\lambda} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \sin \frac{2n+1}{3} \pi \left(\frac{\lambda}{\lambda^2 + k^2} \right) \int_{-\infty}^0 e^{-\lambda \eta} J_{\frac{2n+1}{3}}(|\eta| k \sin a) d\eta$$

$$\left. - \frac{8}{3} \Phi_0 k \sin a \frac{ik \cos a}{\lambda} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \sin \frac{2}{3} (n-1) \pi \left(\frac{\lambda}{\lambda^2 + k^2} \right) \int_{-\infty}^0 e^{-\lambda \eta} J_{\frac{2}{3}}(n-1) (|\eta| k \sin a) d\eta \right]$$

or

$$(7.27) \quad c_1 A_2 + c_2 B_2 = D_2.$$

We shall now evaluate the integrals appearing in (7.16) and (7.26).

If we make the substitution $\eta = -\zeta$ in both expressions and let

$$(7.28) \quad \lambda = \frac{ik}{s}$$

$$(7.29) \quad \left\{ \begin{array}{l} I_v^{(0)} = \int_0^\infty J_v(\zeta k \sin \alpha) e^{-\frac{k^2}{\lambda} \zeta} d\zeta \\ I_v^{(1)} = \int_0^\infty H_v^{(1)}(\zeta k \sin \alpha) e^{-\frac{k^2}{\lambda} \zeta} d\zeta \\ L_v^{(0)} = \int_0^\infty J_v(\zeta k \sin \alpha) e^{+\lambda \zeta} d\zeta \\ L_v^{(1)} = \int_0^\infty H_v^{(1)}(\zeta k \sin \alpha) e^{+\lambda \zeta} d\zeta \end{array} \right.$$

equations (7.16) and (7.26) become

$$(7.30) \quad c_1 \left[+s \sin \frac{\pi}{3} I_{\frac{1}{3}}^{(1)} + \sin \alpha e^{i\frac{\pi}{6}} \sin \frac{2\pi}{3} I_{\frac{2}{3}}^{(1)} \right] + c_2 \left[\cos \alpha \sin \frac{2\pi}{3} I_{\frac{2}{3}}^{(1)} \right] \\ = \left[-\frac{8}{3} \Phi_0 s \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \sin \frac{2}{3} (n+1) \pi I_{\frac{2n+1}{3}}^{(0)} \right. \\ + \frac{8}{3} \Phi_0 \sin \alpha e^{-i\frac{\pi}{2}} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \sin \frac{2}{3} (n-1) \pi I_{\frac{2n+1}{3}}^{(0)} \\ \left. - \frac{8}{3} \Phi_0 \cos \alpha \sum_{n=1}^{\infty} \exp \left[-i \left(\frac{2n}{3} \right) \frac{\pi}{2} \right] \sin \frac{2n}{3} \theta_0 \sin \frac{2n}{3} \pi I_{\frac{2n}{3}}^{(0)} \right]$$

and

$$\begin{aligned}
 (7.31) \quad & c_1 \left[+s \cos \alpha \sin \frac{\pi}{3} L_1^{(1)} + s^2 \cos \alpha \sin \alpha e^{i\frac{\pi}{6}} \sin \frac{2}{3} \pi L_2^{(1)} \right] \\
 & + c_2 \left[(1-s^2 \sin^2 \alpha) \sin \frac{2}{3} \pi L_2^{(1)} \right] \\
 & = \left[-\frac{8}{3} \mathbb{I}_0 s \cos \alpha \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \sin \frac{2n+1}{3} \pi L_{\frac{2n+1}{3}}^{(0)} \right. \\
 & \quad + \frac{8}{3} \mathbb{I}_0 s^2 \cos \alpha \sin \alpha e^{-i\frac{\pi}{2}} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \sin \frac{2}{3} (n-1) L_{\frac{2}{3}(n-1)}^{(0)} \\
 & \quad \left. - \frac{8}{3} \mathbb{I}_0 (1-s^2 \sin^2 \alpha) \sum_{n=1}^{\infty} \exp \left[-i \left(\frac{2n}{3} \right) \frac{\pi}{2} \right] \sin \frac{2n}{3} \theta_0 \sin \frac{2n}{3} \pi L_{\frac{2n}{3}}^{(0)} \right].
 \end{aligned}$$

The last two equations can be written in a much more compact form if we introduce the following quantities:

$$(7.32) \quad \cos \theta_{s_1} = \frac{1}{s \sin \alpha}$$

$$(7.33) \quad \cos \theta_{s_2} = \frac{s}{\sin \alpha}$$

$$(7.34) \quad \begin{cases} R(\theta_{s_1}) = \left\{ \sin \frac{\pi}{3} L_1^{(1)} + \frac{1}{\cos \theta_{s_1}} e^{i\frac{\pi}{6}} \sin \frac{2\pi}{3} L_2^{(1)} \right\} \\ R(\theta_{s_2}) = \left\{ \sin \frac{\pi}{3} I_1^{(1)} + \frac{1}{\cos \theta_{s_2}} e^{i\frac{\pi}{6}} \sin \frac{2\pi}{3} I_2^{(1)} \right\} \end{cases}$$

$$(7.35) \left\{ \begin{aligned} S(\theta_{s_1}) &= \left\{ \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \sin \frac{2n+1}{3} \pi L_{\frac{2n+1}{3}}^{(0)} \right. \\ &\quad \left. - \frac{e^{-i \frac{\pi}{2}}}{\cos \theta_{s_2}} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \sin \frac{2}{3}(n-1)\pi L_{\frac{2}{3}(n-1)}^{(0)} \right\} \\ S(\theta_{s_2}) &= \left\{ \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \sin \frac{2n+1}{3} \pi I_{\frac{2n+1}{3}}^{(0)} \right. \\ &\quad \left. - \frac{e^{-i \frac{\pi}{2}}}{\cos \theta_{s_2}} \sum_{n=0}^{\infty} \exp \left[-i \left(\frac{2n+1}{3} \right) \frac{\pi}{2} \right] \cos \frac{2n+1}{3} \theta_0 \sin \frac{2}{3}(n-1)\pi I_{\frac{2}{3}(n-1)}^{(0)} \right\} \end{aligned} \right.$$

$$(7.36) \left\{ \begin{aligned} T(\theta_{s_1}) &= \left\{ \sum_{n=1}^{\infty} \exp \left[-i \left(\frac{2n}{3} \right) \frac{\pi}{2} \right] \sin \frac{2n}{3} \theta_0 \sin \frac{2n}{3} \pi L_{\frac{2n}{3}}^{(0)} \right\} \\ T(\theta_{s_2}) &= \left\{ \sum_{n=1}^{\infty} \exp \left[-i \left(\frac{2n}{3} \right) \frac{\pi}{2} \right] \sin \frac{2n}{3} \theta_0 \sin \frac{2n}{3} \pi I_{\frac{2n}{3}}^{(0)} \right\} . \end{aligned} \right.$$

If we make the above substitutions into (7.16) and (7.26), we obtain

$$(7.37) \quad c_1 s R(\theta_{s_2}) + c_2 \cos \alpha \sin \frac{2\pi}{3} I_{\frac{2}{3}}^{(1)} = -\frac{8}{3} \bar{\Psi}_0 s S(\theta_{s_2}) - \frac{8}{3} \bar{\Psi}_0 \cos \alpha T(\theta_{s_2})$$

and

$$(7.38) \quad c_1 s \cos \alpha R(\theta_{s_1}) + c_2 (1-s^2 \sin^2 \alpha) \sin \frac{2\pi}{3} L_{\frac{2}{3}}^{(1)} \\ = -\frac{8}{3} \bar{\Psi}_0 s \cos \alpha S(\theta_{s_1}) - \frac{8}{3} \bar{\Psi}_0 (1-s^2 \sin^2 \alpha) T(\theta_{s_1}),$$

respectively.

The integral expressions defined in (7.29) can be evaluated explicitly using well-known results for the Bessel and Hankel functions. In terms of the notation of equations (7.28), (7.32) and (7.33), the values of the integrals turn out to be

$$(7.39) \quad \left\{ \begin{array}{l} I_v^{(0)} = \frac{\exp\left[-iv\left(\theta_{s_2} - \frac{\pi}{2}\right)\right]}{k \sin \alpha \sin \theta_{s_2}} \\ I_v^{(1)} = \frac{2}{\sin v\pi} \frac{e^{-iv\frac{\pi}{2}}}{k \sin \alpha \sin \theta_{s_2}} \sin v \theta_{s_2} \\ L_v^{(0)} = \frac{\exp\left[-iv\left(\theta_{s_1} - \frac{\pi}{2}\right)\right]}{k \sin \alpha \sin \theta_{s_1}} \\ L_v^{(1)} = \frac{2}{\sin v\pi} \frac{e^{-iv\frac{\pi}{2}}}{k \sin \alpha \sin \theta_{s_1}} \sin v \theta_{s_1} \end{array} \right. .$$

We note that in terms of θ_{s_1} and θ_{s_2} , the quantities $I_v^{(i)}$ and $L_v^{(i)}$ are formally identical. Consequently, if we evaluate $R(\theta_{s_2})$, $S(\theta_{s_2})$ and $T(\theta_{s_2})$, then we can determine $R(\theta_{s_1})$, $S(\theta_{s_1})$ and $T(\theta_{s_1})$ by merely replacing θ_{s_2} by θ_{s_1} . We now proceed to evaluate $R(\theta_{s_2})$, $S(\theta_{s_2})$ and $T(\theta_{s_2})$. Consider first $R(\theta_{s_2})$. Substituting for $I_{\frac{1}{3}}^{(1)}$ and $I_{\frac{2}{3}}^{(1)}$ from (7.39) we find that

$$(7.40) \quad R(\theta_{s_2}) = \frac{2e^{-i\frac{\pi}{6}} \cos \frac{\theta_{s_2}}{3}}{k \sin \alpha \cos \theta_{s_2}} .$$

Consider next $T(\theta_{s_2})$. Substituting for $I_{\frac{2n}{3}}^{(0)}$ and writing all summations in terms of exponentials, we have

$$(7.41) \quad T(\theta_{s_2}) = \frac{-1}{4k \sin \alpha \sin \theta_{s_2}} \sum_{n=1}^{\infty} \left[\exp \left[i \frac{2n}{3} (\theta_0 + \pi - \theta_{s_2}) \right] - \exp \left[i \frac{2n}{3} (\theta_0 - \pi - \theta_{s_2}) \right] \right. \\ \left. - \exp \left[-i \frac{2n}{3} (\theta_0 - \pi + \theta_{s_2}) \right] + \exp \left[-i \frac{2n}{3} (\theta_0 + \pi + \theta_{s_2}) \right] \right]$$

Summing the above geometrical series, we obtain

$$(7.42) \quad T(\theta_{s_2}) = \frac{-1}{4k \sin \alpha \sin \theta_{s_2}} \left[\frac{\exp \left[i \frac{2}{3} (\theta_0 + \pi - \theta_{s_2}) \right]}{1 - \exp \left[i \frac{2}{3} (\theta_0 + \pi - \theta_{s_2}) \right]} - \frac{\exp \left[i \frac{2}{3} (\theta_0 - \pi - \theta_{s_2}) \right]}{1 - \exp \left[i \frac{2}{3} (\theta_0 - \pi - \theta_{s_2}) \right]} \right. \\ \left. - \frac{\exp \left[-i \frac{2}{3} (\theta_0 - \pi + \theta_{s_2}) \right]}{1 - \exp \left[-i \frac{2}{3} (\theta_0 - \pi + \theta_{s_2}) \right]} + \frac{\exp \left[-i \frac{2}{3} (\theta_0 + \pi + \theta_{s_2}) \right]}{1 - \exp \left[-i \frac{2}{3} (\theta_0 + \pi + \theta_{s_2}) \right]} \right].$$

By further manipulation and use of certain trigonometrical identities, we have

$$(7.43) \quad T(\theta_{s_2}) = \frac{-i \sqrt{3}}{4k \sin \alpha \sin \theta_{s_2}} \left[\frac{\sin \frac{1}{3} (\theta_0 - \theta_{s_2})}{\sin (\theta_0 - \theta_{s_2})} - \frac{\sin \frac{1}{3} (\theta_0 + \theta_{s_2})}{\sin (\theta_0 + \theta_{s_2})} \right].$$

Consider, finally, $S(\theta_{s_2})$. The evaluation of this expression is considerably

more complicated than that for $T(\theta_{s_2})$, although the procedure is similar.

Substituting for $I_{\frac{2n+1}{3}}^{(o)}$ and $I_{\frac{2}{3}(n-1)}^{(o)}$ and writing all summations in terms of exponentials, we have

$$\begin{aligned}
 (7.44) \quad S(\theta_{s_2}) &= \frac{-1}{\mu k s_1 n \sin \theta_{s_2}} \sum_{n=0}^{\infty} \left[\exp \left[i \frac{2n+1}{3} (\theta_0 + \pi - \theta_{s_2}) \right] - \exp \left[i \frac{2n+1}{3} (\theta_0 - \pi - \theta_{s_2}) \right] + \exp \left[-i \frac{2n+1}{3} (\theta_0 - \pi + \theta_{s_2}) \right] \right. \\
 &\quad \left. - \exp \left[-i \frac{2n+1}{3} (\theta_0 + \pi + \theta_{s_2}) \right] \right] \\
 &+ \frac{-1 \cos \theta_0}{\mu k s_1 n \sin \theta_{s_2} \cos \theta_{s_2}} \sum_{n=0}^{\infty} \left[\exp \left[i \frac{2}{3} (n-1) (\theta_0 + \pi - \theta_{s_2}) \right] + \exp \left[-i \frac{2}{3} (n-1) (\theta_0 - \pi + \theta_{s_2}) \right] \right. \\
 &\quad \left. - \exp \left[i \frac{2}{3} (n-1) (\theta_0 - \pi - \theta_{s_2}) \right] - \exp \left[-i \frac{2}{3} (n-1) (\theta_0 + \pi + \theta_{s_2}) \right] \right] \\
 &+ \frac{\sin \theta_0}{\mu k s_1 n \sin \theta_{s_2} \cos \theta_{s_2}} \sum_{n=0}^{\infty} \left[\exp \left[i \frac{2}{3} (n-1) (\theta_0 + \pi - \theta_{s_2}) \right] - \exp \left[-i \frac{2}{3} (n-1) (\theta_0 - \pi + \theta_{s_2}) \right] \right. \\
 &\quad \left. - \exp \left[i \frac{2}{3} (n-1) (\theta_0 - \pi - \theta_{s_2}) \right] + \exp \left[-i \frac{2}{3} (n-1) (\theta_0 + \pi + \theta_{s_2}) \right] \right] \cdot
 \end{aligned}$$

Summing the above geometrical series, we obtain

$$\begin{aligned}
 (7.45) \quad S(\theta_{s_2}) = & \frac{-1}{\mu k s_1 n a \sin \theta_{s_2}} \left[\frac{\exp \left[i \frac{1}{3} (\theta_0 + \pi - \theta_{s_2}) \right]}{1 - \exp \left[i \frac{2}{3} (\theta_0 + \pi - \theta_{s_2}) \right]} - \frac{\exp \left[i \frac{1}{3} (\theta_0 - \pi - \theta_{s_2}) \right]}{1 - \exp \left[i \frac{2}{3} (\theta_0 - \pi - \theta_{s_2}) \right]} + \frac{\exp \left[-i \frac{1}{3} (\theta_0 - \pi + \theta_{s_2}) \right]}{1 - \exp \left[-i \frac{2}{3} (\theta_0 - \pi + \theta_{s_2}) \right]} \right. \\
 & \left. - \frac{\exp \left[-i \frac{1}{3} (\theta_0 + \pi + \theta_{s_2}) \right]}{1 - \exp \left[-i \frac{2}{3} (\theta_0 + \pi + \theta_{s_2}) \right]} \right] \\
 & + \frac{-i \cos \theta_0}{\mu k s_1 n a \sin \theta_{s_2} \cos \theta_{s_2}} \left[\frac{\exp \left[-i \frac{2}{3} (\theta_0 + \pi - \theta_{s_2}) \right]}{1 - \exp \left[i \frac{2}{3} (\theta_0 + \pi - \theta_{s_2}) \right]} + \frac{\exp \left[i \frac{2}{3} (\theta_0 - \pi + \theta_{s_2}) \right]}{1 - \exp \left[-i \frac{2}{3} (\theta_0 - \pi + \theta_{s_2}) \right]} - \frac{\exp \left[-i \frac{2}{3} (\theta_0 - \pi - \theta_{s_2}) \right]}{1 - \exp \left[i \frac{2}{3} (\theta_0 - \pi - \theta_{s_2}) \right]} \right. \\
 & \left. - \frac{\exp \left[i \frac{2}{3} (\theta_0 + \pi + \theta_{s_2}) \right]}{1 - \exp \left[-i \frac{2}{3} (\theta_0 + \pi + \theta_{s_2}) \right]} \right] \\
 & + \frac{\sin \theta_0}{\mu k s_1 n a \sin \theta_{s_2} \cos \theta_{s_2}} \left[\frac{\exp \left[-i \frac{2}{3} (\theta_0 + \pi - \theta_{s_2}) \right]}{1 - \exp \left[i \frac{2}{3} (\theta_0 + \pi - \theta_{s_2}) \right]} - \frac{\exp \left[+i \frac{2}{3} (\theta_0 - \pi + \theta_{s_2}) \right]}{1 - \exp \left[-i \frac{2}{3} (\theta_0 - \pi + \theta_{s_2}) \right]} - \frac{\exp \left[-i \frac{2}{3} (\theta_0 - \pi - \theta_{s_2}) \right]}{1 - \exp \left[i \frac{2}{3} (\theta_0 - \pi - \theta_{s_2}) \right]} \right. \\
 & \left. + \frac{\exp \left[+i \frac{2}{3} (\theta_0 + \pi + \theta_{s_2}) \right]}{1 - \exp \left[-i \frac{2}{3} (\theta_0 + \pi + \theta_{s_2}) \right]} \right]
 \end{aligned}$$

By proper combination of the above terms it is possible, after considerable manipulation, to obtain the following simple expression:

$$(7.46) \quad S(\theta_{s_2}) = \frac{-i\sqrt{3}}{4ksin\alpha\cos\theta_{s_2}} \left[\frac{\sin \frac{2}{3} (\theta_0 - \theta_{s_2})}{\sin(\theta_0 - \theta_{s_2})} + \frac{\sin \frac{2}{3} (\theta_0 + \theta_{s_2})}{\sin(\theta_0 + \theta_{s_2})} \right] .$$

If we make the following substitutions

$$(7.47) \quad E(\theta, \theta_0) = \frac{1}{\sqrt{3}} \left(\frac{\sin \frac{2}{3} (\theta_0 - \theta)}{\sin (\theta_0 - \theta)} + \frac{\sin \frac{2}{3} (\theta_0 + \theta)}{\sin (\theta_0 + \theta)} \right)$$

$$(7.48) \quad F(\theta, \theta_0) = \frac{1}{\sqrt{3}} \left(\frac{\sin \frac{1}{3} (\theta_0 - \theta)}{\sin (\theta_0 - \theta)} - \frac{\sin \frac{1}{3} (\theta_0 + \theta)}{\sin (\theta_0 + \theta)} \right) ,$$

the expressions given by (7.37) and (7.38) finally reduce to

$$(7.49) \quad c_1 e^{i\frac{\pi}{3}} \sin \alpha \sin \theta_{s_2} \cos \frac{\theta_{s_2}}{3} + c_2 e^{i\frac{\pi}{6}} \cos \alpha \sin \frac{2}{3} \theta_{s_2} \\ = - \bar{\Psi}_0 E(\theta_{s_2}, \theta_0) \sin \alpha \sin \theta_{s_2} - \bar{\Psi}_0 F(\theta_{s_2}, \theta_0) \cos \alpha$$

and

$$(7.50) \quad c_1 e^{i\frac{\pi}{3}} \cos \alpha \cos \frac{\theta_{s_1}}{3} - c_2 e^{i\frac{\pi}{6}} \sin \alpha \sin \theta_{s_1} \sin \frac{2}{3} \theta_{s_1} \\ = - \bar{\Psi}_0 E(\theta_{s_1}, \theta_0) \cos \alpha + \bar{\Psi}_0 F(\theta_{s_1}, \theta_0) \sin \alpha \sin \theta_{s_1} .$$

Solving for the desired constants c_1 and c_2 , we obtain

$$(7.51) \quad c_1 e^{i\frac{\pi}{3}} = \frac{-1}{\Delta} \left[\left\{ \mathbb{I}_O^E(\theta_{s_2}, \theta_o) \sin \alpha \sin \theta_{s_2} + \mathbb{I}_O^F(\theta_{s_2}, \theta_o) \cos \alpha \right\} \sin \alpha \sin \theta_{s_1} \sin \frac{2}{3} \theta_{s_1} \right. \\ \left. + \left\{ \mathbb{I}_O^E(\theta_{s_1}, \theta_o) \cos \alpha - \mathbb{I}_O^F(\theta_{s_1}, \theta_o) \sin \alpha \sin \theta_{s_1} \right\} \cos \alpha \sin \frac{2}{3} \theta_{s_2} \right]$$

$$(7.52) \quad c_2 e^{i\frac{\pi}{6}} = \frac{-1}{\Delta} \left[\left\{ \mathbb{I}_O^E(\theta_{s_2}, \theta_o) \sin \alpha \sin \theta_{s_2} + \mathbb{I}_O^F(\theta_{s_2}, \theta_o) \cos \alpha \right\} \cos \alpha \cos \frac{\theta_{s_1}}{3} \right. \\ \left. - \left\{ \mathbb{I}_O^E(\theta_{s_1}, \theta_o) \cos \alpha - \mathbb{I}_O^F(\theta_{s_1}, \theta_o) \sin \alpha \sin \theta_{s_1} \right\} \sin \alpha \sin \theta_{s_2} \cos \frac{\theta_{s_2}}{3} \right]$$

where

$$(7.53) \quad \Delta = \left\{ \sin^2 \alpha \sin \theta_{s_1} \sin \frac{2}{3} \theta_{s_1} \sin \theta_{s_2} \cos \frac{\theta_{s_2}}{3} + \cos^2 \alpha \cos \frac{\theta_{s_1}}{3} \sin \frac{2}{3} \theta_{s_2} \right\}$$

and

$$(7.54) \quad \sin \theta_{s_{1,2}} = +1 \sqrt{\cos^2 \theta_{s_{1,2}} - 1}.$$

The definitions for $\cos \theta_{s_{1,2}}$ are given by (7.32) and (7.33), respectively.

We note that θ_{s_1} and θ_{s_2} are positive imaginary in the range of s and $\sin \alpha$

in which our evaluation has been carried out by quadratures. When the

formulas are applied for ranges of s and $\sin \alpha$ in which the angles θ_{s_1} and

θ_{s_2} are real, an ambiguity arises as to the sign of the angles. But it

is to be noted that the values of the constants c_1 and c_2 are unaffected

by a change in sign of either (or both) θ_{s_1} and θ_{s_2} .

The values of the c 's given above are valid for the entire range of

$\sin \alpha$ and s . However, it should be noted that the actual calculation of the

c 's was based on a restricted range given by the inequalities $s > \sin \alpha$ and

$\frac{1}{s} > \sin \alpha$. We shall now explain this circumstance. It is easily seen that

the c's are obtained by solving a pair of linear equations of the form $K_1 c_1 + L_1 c_2 = M_1$ and $K_2 c_1 + L_2 c_2 = M_2$, where the terms K_i, L_i and M_i involve integrals of ϕ_S, ϕ_F, Ψ_S and Ψ_F . A typical term, for example, is

$$\int_{-\infty}^0 e^{-\lambda \xi} \phi_F d\xi.$$

Although this integral exists in the full range, as explained earlier, the terms

$$\int_{-\infty}^0 e^{-\lambda \xi} J_{\frac{2n}{3}}(|\xi| k \sin \alpha) d\xi,$$

which result from replacing ϕ_F by an infinite series, converge only in the restricted range. With the determination of the constants c_1 and c_2 we have solved completely the diffraction problem stated in Section 2. The formal exact solutions for the electric fields E_x, E_y and E_z are given by (7.1), (7.2) and (7.3) respectively. The quantities $\phi(x,y)$ and $\Psi(x,y)$, which appear in the expressions for the electric fields, are defined in (7.4) and (7.5) and also in (7.6) through (7.9). The constants c_1 and c_2 are given by (7.51) and (7.52).

8. Far field evaluation of the original electric fields

Although we have obtained an exact solution for the diffraction problem, it is desirable to find a simple expression for the far field. The transformation equations are given by

$$(8.1) \quad \begin{aligned} \zeta_y &= -\frac{ik}{s} E_y + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \\ \zeta_z &= -\frac{ik}{s} E_z - \frac{\partial E_x}{\partial z} + \frac{\partial E_z}{\partial x} \end{aligned}$$

where we have expressed λ in terms of s and k . The original electric field components in the far field can be written as the sum of two terms. The first term can be found by using geometrical optics; the second is due to the diffracted field. We have

$$(8.2) \quad \begin{cases} E_x = (E_x)_{\text{geom}} + (E_x)_{\text{diff}} \\ E_y = (E_y)_{\text{geom}} + (E_y)_{\text{diff}} \\ E_z = (E_z)_{\text{geom}} + (E_z)_{\text{diff}} \end{cases}$$

where $(E_x)_{\text{geom}}$, $(E_y)_{\text{geom}}$ and $(E_z)_{\text{geom}}$ are assumed to be known, and $(E_x)_{\text{diff}}$, $(E_y)_{\text{diff}}$ and $(E_z)_{\text{diff}}$ are to be found. The transformed electric field components ζ_y and ζ_z in the far field can be written in the form

$$(8.3) \quad \zeta_y = (\zeta_y)_{\text{geom}} + [(\zeta_y)_{\text{diff}}]_F + c_1 [(\zeta_y)_{\text{diff}}]_S$$

$$(8.4) \quad \zeta_z = (\zeta_z)_{\text{geom}} + [(\zeta_z)_{\text{diff}}]_F + c_2 [(\zeta_z)_{\text{diff}}]_S$$

The far field solutions of the transformed problems which are finite at the edge of the wedge, and are denoted by the subscript F in (8.3) and (8.4), can be obtained from the paper of Reiche^[5]. Reiche uses the method of reflection on Riemann sheets due to Sommerfeld^[7] and applies it to the case of a plane

wave diffracted by a right-angled wedge where the boundary conditions require the vanishing of the field on the surface. By using some of Reiche's intermediate results, it is easy to solve the problem of the diffraction of a plane wave by a right-angled wedge where the boundary conditions require the vanishing of the field on one surface ($\theta = \frac{3\pi}{2}$) and the vanishing of the derivative on the other ($\theta = 0$). We note that the only required changes in notation between the results of Reiche and those used here are the following: (1) replace k by $k \sin \alpha$, (2) replace i by $-i$, since we use the time factor $e^{-i\omega t}$, and (3) multiply the solution by $\bar{\Phi}_0$ or $\bar{\Psi}_0$ in order to have the proper amplitude. The far field solutions of the transformed problems which are singular at the edge of the wedge, and are denoted by the subscript S in (8.3) and (8.4), are the asymptotic forms of equations (7.7) and (7.9). The constants c_1 and c_2 are the same constants as those appearing in the exact solution and are given by (7.51) and (7.52). The far field expressions for the finite and singular solutions are given by

$$(8.5) \quad [(\xi_y)_{\text{diff}}]_F = - \frac{2 \bar{\Phi}_0 e^{i\frac{\pi}{4}}}{\sqrt{6\pi k r \sin \alpha}} \left(\frac{\sin^2(\theta_0 - \theta)}{\sin(\theta_0 - \theta)} + \frac{\sin^2(\theta_0 + \theta)}{\sin(\theta_0 + \theta)} \right) e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.6) \quad [(\xi_z)_{\text{diff}}]_F = - \frac{2 \bar{\Psi}_0 e^{i\frac{\pi}{4}}}{\sqrt{6\pi k r \sin \alpha}} \left(\frac{\sin^{\frac{1}{3}}(\theta_0 - \theta)}{\sin(\theta_0 - \theta)} - \frac{\sin^{\frac{1}{3}}(\theta_0 + \theta)}{\sin(\theta_0 + \theta)} \right) e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.7) \quad [(\xi_y)_{\text{diff}}]_S = \sqrt{\frac{2}{\pi k r \sin \alpha}} e^{-i\frac{5\pi}{12}} \cos \frac{1}{3} \theta e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.8) \quad [(\xi_z)_{\text{diff}}]_S = \sqrt{\frac{2}{\pi k r \sin \alpha}} e^{-i\frac{7\pi}{12}} \sin \frac{2}{3} \theta e^{ikr \sin \alpha} e^{ikz \cos \alpha} .$$

If we use (7.47) and (7.48), the finite transformed far fields simplify to

$$(8.9) \quad \left[(\xi_y)_{\text{diff}} \right]_F = -\sqrt{\frac{2}{\pi k r \sin \alpha}} e^{+i\frac{\pi}{4}} \Phi_0^F(\theta, \theta_0) e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.10) \quad \left[(\xi_z)_{\text{diff}} \right]_F = -\sqrt{\frac{2}{\pi k r \sin \alpha}} e^{+i\frac{\pi}{4}} \Psi_0^F(\theta, \theta_0) e^{ikr \sin \alpha} e^{ikz \cos \alpha} .$$

We expect the far field forms of the original electric fields $(E_x)_{\text{diff}}$, $(E_y)_{\text{diff}}$ and $(E_z)_{\text{diff}}$ to have forms similar to that given by (8.9) and (8.10). Therefore, we write

$$(8.11) \quad (E_x)_{\text{diff}} = \frac{\ell(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.12) \quad (E_y)_{\text{diff}} = \frac{m(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.13) \quad (E_z)_{\text{diff}} = \frac{n(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha} .$$

The quantities $\ell(\theta)$, $m(\theta)$ and $n(\theta)$ are as yet unknown. Consider now only the diffracted part of the far field expressions. The transformation equations become

$$(8.14) \quad (\xi_y)_{\text{diff}} = -\frac{ik}{s} (E_y)_{\text{diff}} + \frac{\partial}{\partial x} (E_y)_{\text{diff}} - \frac{\partial}{\partial y} (E_x)_{\text{diff}}$$

$$(8.15) \quad (\xi_z)_{\text{diff}} = -\frac{ik}{s} (E_z)_{\text{diff}} - \frac{\partial}{\partial z} (E_x)_{\text{diff}} + \frac{\partial}{\partial x} (E_z)_{\text{diff}} .$$

In the far field we note that

$$(8.16) \quad \left\{ \begin{array}{l} \frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \rightarrow \cos \theta \frac{\partial}{\partial r} \\ \frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \frac{1}{r} \cos \theta \frac{\partial}{\partial \theta} \rightarrow \sin \theta \frac{\partial}{\partial r} \\ \frac{\partial}{\partial z} = \frac{\partial}{\partial z} . \end{array} \right.$$

Therefore,

$$(8.17) \quad (\mathcal{E}_y)_{\text{diff}} = -\frac{ik}{s} (E_y)_{\text{diff}} + \cos \theta \frac{\partial}{\partial r} (E_y)_{\text{diff}} - \sin \theta \frac{\partial}{\partial r} (E_x)_{\text{diff}}$$

$$(8.18) \quad (\mathcal{E}_z)_{\text{diff}} = -\frac{ik}{s} (E_z)_{\text{diff}} - \frac{\partial}{\partial z} (E_x)_{\text{diff}} + \cos \theta \frac{\partial}{\partial r} (E_z)_{\text{diff}} .$$

Substituting for the transformed and original electric fields into (8.17) and (8.18), we obtain

$$(8.19) \quad -\sqrt{\frac{2}{\pi k s \sin \alpha}} e^{+i\frac{\pi}{4}} \mathcal{A} = (\cos \theta \sin \alpha - \frac{1}{s})m(\theta) - \sin \theta \sin \alpha \ell(\theta)$$

and

$$(8.20) \quad -\sqrt{\frac{2}{\pi k s \sin \alpha}} e^{+i\frac{\pi}{4}} \mathcal{B} = (\cos \theta \sin \alpha - \frac{1}{s})n(\theta) - \cos \alpha \ell(\theta)$$

where

$$(8.21) \quad \mathcal{A} = \frac{1}{ik} \left[\mathcal{I}_0 F(\theta, \theta_0) + c_1 e^{+i\frac{\pi}{3}} \cos \frac{1}{3} \theta \right]$$

$$(8.22) \quad \mathcal{B} = \frac{1}{ik} \left[\mathcal{I}_0 F(\theta, \theta_0) + c_2 e^{+i\frac{\pi}{6}} \sin \frac{2}{3} \theta \right] .$$

We now have two equations for the three unknowns $\ell(\theta)$, $m(\theta)$ and $n(\theta)$. Another equation can be found by using the divergence condition since $\ell(\theta)$, $m(\theta)$ and $n(\theta)$ are not all independent. We must first, however, express the divergence condition for the rectangular components of the diffracted field in terms of cylindrical coordinates. If we substitute, using (8.11), (8.12) and (8.13), and keep terms of order $\frac{1}{\sqrt{r}}$ only, the divergence condition becomes

$$(8.23) \quad \cos \theta \sin \alpha \ell(\theta) + \sin \theta \sin \alpha m(\theta) + \cos \alpha n(\theta) = 0.$$

This gives us the desired third equation. Solving for $\ell(\theta)$, $m(\theta)$ and $n(\theta)$, we obtain

$$(8.24) \quad \ell(\theta) = -e^{i\frac{\pi}{4}} \sqrt{\frac{2}{\pi k \sin \alpha}} \frac{1}{D} \left\{ A \sin \theta \sin \alpha + B \cos \alpha \right\} \left\{ s \cos \theta \sin \alpha - 1 \right\}$$

$$(8.25) \quad m(\theta) = -e^{i\frac{\pi}{4}} \sqrt{\frac{2}{\pi k \sin \alpha}} \frac{s}{D} \left[A \left\{ \left(\frac{1}{s} \cos \theta \sin \alpha - 1 \right) + \sin^2 \alpha \sin^2 \theta \right\} + B \cos \alpha \sin \alpha \sin \theta \right]$$

$$(8.26) \quad n(\theta) = -e^{i\frac{\pi}{4}} \sqrt{\frac{2}{\pi k \sin \alpha}} \frac{s}{D} \left[A \cos \alpha \sin \alpha \sin \theta + B \left\{ \left(\frac{1}{s} \cos \theta \sin \alpha - 1 \right) + \cos^2 \alpha \right\} \right]$$

where

$$(8.27) \quad D = \left(\frac{1}{s} \cos \theta \sin \alpha - 1 \right) (s \cos \theta \sin \alpha - 1).$$

It is possible from the above results to verify in a simple way that the constants c_1 and c_2 used in the far field approximation have the precise values already obtained in the exact treatment. We note that D is zero for certain values of the angle θ . Those values of θ which make D zero are given by

$$(8.28) \quad \begin{cases} \cos \theta = \frac{1}{s \sin \alpha} \\ \cos \theta = \frac{s}{\sin \alpha} \end{cases} .$$

These particular angles, however, were defined as θ_{s_1} and θ_{s_2} , respectively, in (7.32) and (7.33). Since the values of s and $\sin \alpha$ are always real and positive, we observe that either θ_{s_1} or θ_{s_2} is real. Consequently, there is always one real angle of incidence for which D is zero. It is easily shown, however, that for any real angle of incidence that makes D zero, our choice of c_1 and c_2 guarantees that the numerators of the ex-

pressions for $m(\theta)$ and $n(\theta)$ vanish to the same order as the denominator D . Consequently, because of our choice of c_1 and c_2 , there are no real angles of incidence other than shadow lines for any choice of s and $\sin \alpha$ for which the far fields are infinite. If we make use of the definitions for θ_{s_1} and θ_{s_2} , the expressions for $l(\theta)$, $m(\theta)$ and $n(\theta)$ become

$$(8.24') \quad l(\theta) = e^{i\frac{5\pi}{4}} \sqrt{\frac{1}{2\pi k \sin \alpha}} \frac{1}{\Delta'} (\cos \theta - \cos \theta_{s_1}) \left[a' \sin \theta + b' \sqrt{\cos \theta_{s_1} \cos \theta_{s_2} - 1} \right]$$

$$(8.25') \quad m(\theta) = e^{i\frac{5\pi}{4}} \sqrt{\frac{1}{2\pi k \sin \alpha}} \frac{1}{\Delta'} \left[a' \left\{ \cos \theta_{s_1} (\cos \theta - \cos \theta_{s_2}) + \sin^2 \theta \right\} \right. \\ \left. + b' \sin \theta \sqrt{\cos \theta_{s_1} \cos \theta_{s_2} - 1} \right]$$

$$(8.26') \quad n(\theta) = e^{i\frac{5\pi}{4}} \sqrt{\frac{1}{2\pi k \sin \alpha}} \frac{1}{\Delta'} \left[a' \sin \theta \sqrt{\cos \theta_{s_1} \cos \theta_{s_2} - 1} \right. \\ \left. + b' \left\{ \cos \theta \cos \theta_{s_1} - 1 \right\} \right]$$

where

$$(7.53') \quad \Delta' = \cos \frac{\theta_{s_1}}{3} \cos \frac{\theta_{s_2}}{3} (\cos \theta - \cos \theta_{s_1}) (\cos \theta - \cos \theta_{s_2}) \\ \cdot \left[\sin \frac{\theta_{s_1}}{3} (\sin \theta_{s_1} \sin \theta_{s_2}) + \sin \frac{\theta_{s_2}}{3} (\cos \theta_{s_1} \cos \theta_{s_2} - 1) \right]$$

$$(8.21') \quad a' = \Phi'_0 \sin \frac{2}{3} \theta_{s_1} (\sin \theta_{s_1} \sin \theta_{s_2}) \left[E(\theta, \theta_0) \cos \frac{\theta_{s_2}}{3} - E(\theta_{s_2}, \theta_0) \cos \frac{\theta}{3} \right] \\ + \Phi'_0 (\cos \theta_{s_1} \cos \theta_{s_2} - 1) \sin \frac{2}{3} \theta_{s_2} \left[E(\theta, \theta_0) \cos \frac{\theta_{s_1}}{3} - E(\theta_{s_1}, \theta_0) \cos \frac{\theta}{3} \right] \\ + \Psi'_0 \sqrt{\cos \theta_{s_1} \cos \theta_{s_2} - 1} \sin \theta_{s_1} \cos \frac{\theta}{3} \left[F(\theta_{s_1}, \theta_0) \sin \frac{2}{3} \theta_{s_2} - F(\theta_{s_2}, \theta_0) \sin \frac{2}{3} \theta_{s_1} \right]$$

$$\begin{aligned}
 (8.22') \quad \mathcal{B}' = & \Phi_0' \sqrt{\cos\theta_{s_1} \cos\theta_{s_2} - 1} \sin\theta_{s_2} \sin \frac{2\theta}{3} \left[E(\theta_{s_1}, \theta_0) \cos \frac{\theta_{s_2}}{3} - E(\theta_{s_2}, \theta_0) \cos \frac{\theta_{s_1}}{3} \right] \\
 & + \Psi_0' \sin\theta_{s_1} \sin\theta_{s_2} \cos \frac{\theta_{s_2}}{3} \left[F(\theta, \theta_0) \sin \frac{2}{3} \theta_{s_1} - F(\theta_{s_1}, \theta_0) \sin \frac{2}{3} \theta \right] \\
 & + \Psi_0' (\cos\theta_{s_1} \cos\theta_{s_2} - 1) \cos \frac{\theta_{s_1}}{3} \left[F(\theta, \theta_0) \sin \frac{2}{3} \theta_{s_2} - F(\theta_{s_2}, \theta_0) \sin \frac{2}{3} \theta \right]
 \end{aligned}$$

and

$$(4.19') \quad \Phi_0' = E_x^{\text{inc}} \left(\frac{\sin\theta_0}{\cos\theta_{s_1}} \right) - E_y^{\text{inc}} \left(1 + \frac{\cos\theta_0}{\cos\theta_{s_1}} \right)$$

$$(4.20') \quad \Psi_0' = -E_x^{\text{inc}} \left(\frac{\sqrt{\cos\theta_{s_1} \cos\theta_{s_2} - 1}}{\cos\theta_{s_1}} \right) - E_z^{\text{inc}} \left(1 + \frac{\cos\theta_0}{\cos\theta_{s_1}} \right).$$

We shall now summarize the results for the far field. See Figures 3, 4 and 5 for definitions of the regions referred to in Cases A, B and C respectively.

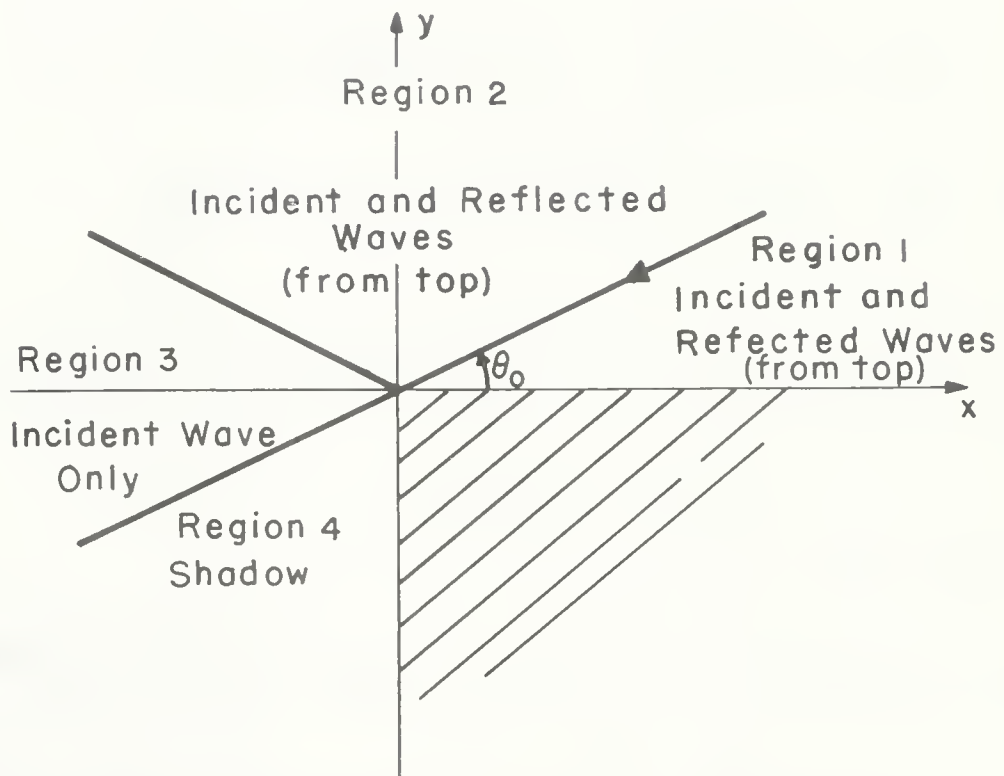
$$\text{Case A: } 0 < \theta_0 < \frac{\pi}{2}$$

1. Region 1 and 2: $(0 < \theta < \pi - \theta_0)$

$$\begin{aligned}
 (8.29) \quad E_x = & E_x^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin\alpha} e^{ikz \cos\alpha} + E_x^{\text{ref}} e^{-ikr \cos(\theta + \theta_0) \sin\alpha} e^{ikz \cos\alpha} \\
 & + \frac{\ell(\theta)}{\sqrt{r}} e^{ikr \sin\alpha} e^{ikz \cos\alpha}
 \end{aligned}$$

$$\begin{aligned}
 (8.30) \quad E_y = & E_y^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin\alpha} e^{ikz \cos\alpha} + E_y^{\text{ref}} e^{-ikr \cos(\theta + \theta_0) \sin\alpha} e^{ikz \cos\alpha} \\
 & + \frac{m(\theta)}{\sqrt{r}} e^{ikr \sin\alpha} e^{ikz \cos\alpha}
 \end{aligned}$$

$$\begin{aligned}
 (8.31) \quad E_z = & E_z^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin\alpha} e^{ikz \cos\alpha} + E_z^{\text{ref}} e^{-ikr \cos(\theta + \theta_0) \sin\alpha} e^{ikz \cos\alpha} \\
 & + \frac{n(\theta)}{\sqrt{r}} e^{ikr \sin\alpha} e^{ikz \cos\alpha}
 \end{aligned}$$



CASE A: $\theta_0 < \frac{\pi}{2}$

Figure 3

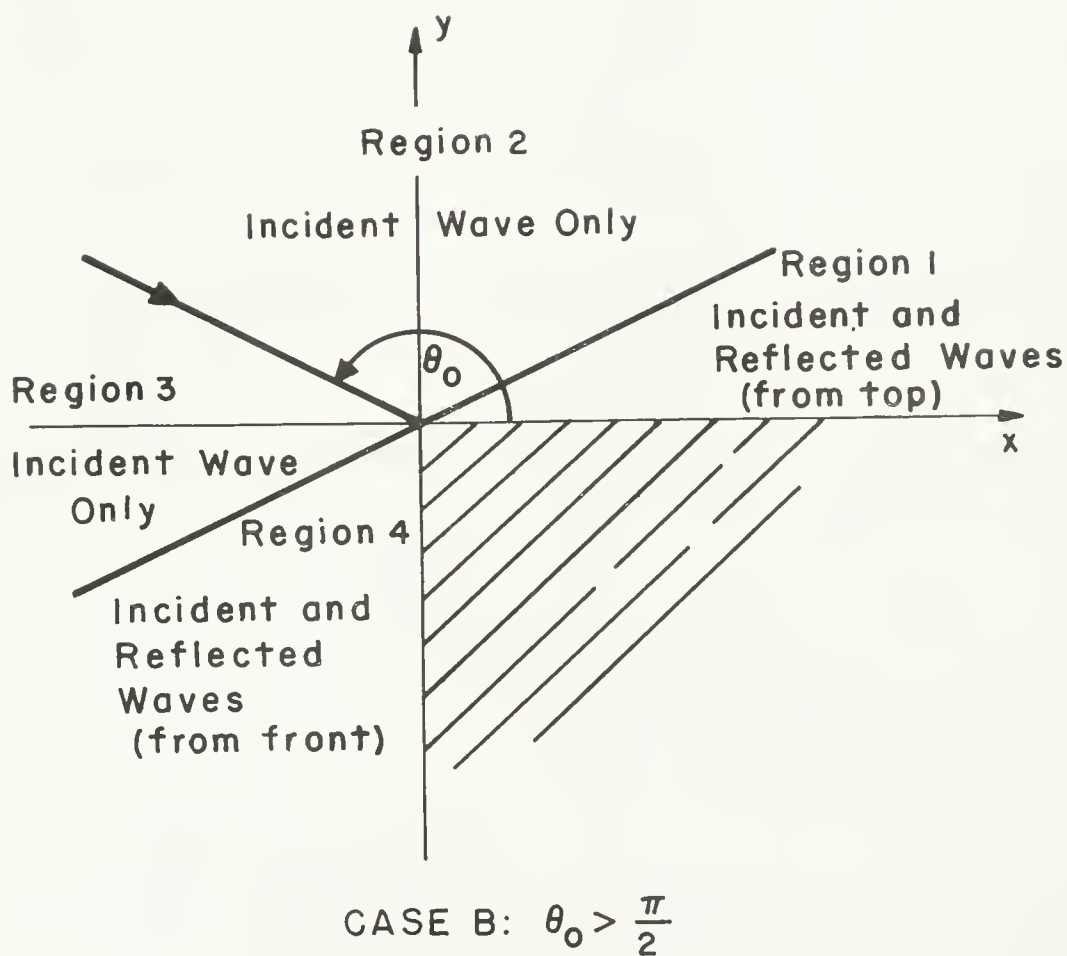


Figure 4

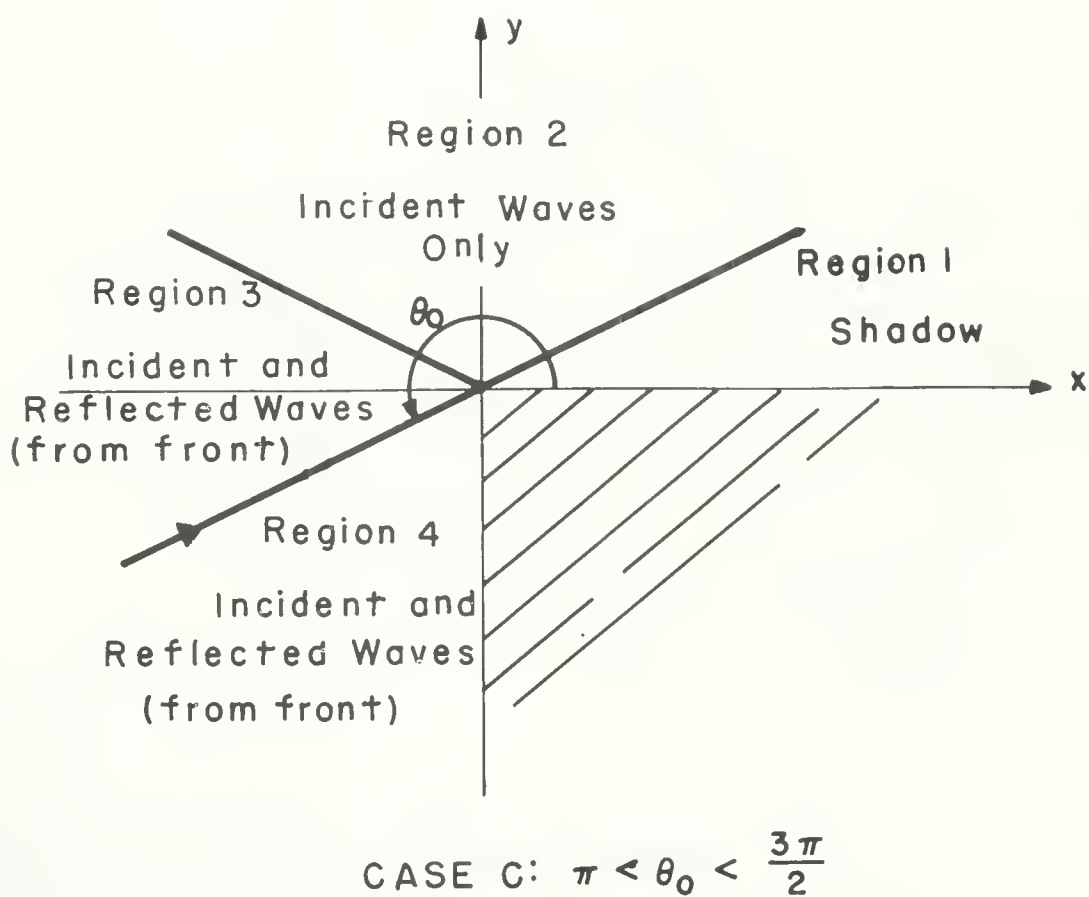


Figure 5

where

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$$(8.32) \quad \begin{cases} E_x^{\text{ref}} = -E_x^{\text{inc}} \\ E_y^{\text{ref}} = \frac{-\sin \alpha \cos \theta_0 E_x^{\text{inc}} + \cos \alpha E_z^{\text{inc}}}{\sin \alpha \sin \theta_0} \\ E_z^{\text{ref}} = -E_z^{\text{inc}} \end{cases} .$$

2. Region 3: $(\pi - \theta_0 < \theta < \pi + \theta_0)$

$$(8.33) \quad E_x = E_x^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin \alpha} e^{ikz \cos \alpha} + \frac{\ell(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.34) \quad E_y = E_y^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin \alpha} e^{ikz \cos \alpha} + \frac{m(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.35) \quad E_z = E_z^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin \alpha} e^{ikz \cos \alpha} + \frac{n(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

3. Region 4: $(\pi + \theta_0 < \theta < \frac{3\pi}{2})$

$$(8.36) \quad E_x = \frac{\ell(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.37) \quad E_y = \frac{m(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.38) \quad E_z = \frac{n(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

Case B: $\frac{\pi}{2} < \theta_0 < \pi$

1. Region 1: $(0 < \theta < \pi - \theta_0)$

$$(8.39) \quad E_x = E_x^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin \alpha} e^{ikz \cos \alpha} + E_x^{\text{ref}} e^{-ikr \cos(\theta + \theta_0) \sin \alpha} e^{ikz \cos \alpha} + \frac{\ell(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.40) \quad E_y = E_y^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin \alpha} e^{ikz \cos \alpha} + E_y^{\text{ref}} e^{-ikr \cos(\theta + \theta_0) \sin \alpha} e^{ikz \cos \alpha} + \frac{m(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.41) \quad E_z = E_z^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin \alpha} e^{ikz \cos \alpha} + E_z^{\text{ref}} e^{-ikr \cos(\theta + \theta_0) \sin \alpha} e^{ikz \cos \alpha} + \frac{n(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

where

$$(8.42) \quad \begin{cases} E_x^{\text{ref}} = -E_x^{\text{inc}} \\ E_y^{\text{ref}} = \frac{-\sin \alpha \cos \theta_0 E_x^{\text{inc}} + \cos \alpha E_z^{\text{inc}}}{\sin \alpha \sin \theta_0} \\ E_z^{\text{ref}} = -E_z^{\text{inc}} \end{cases}$$

2. Regions 2 and 3: $(\pi - \theta_0 < \theta < 2\pi - \theta_0)$

$$(8.43) \quad E_x = E_x^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin \alpha} e^{ikz \cos \alpha} + \frac{l(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.44) \quad E_y = E_y^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin \alpha} e^{ikz \cos \alpha} + \frac{m(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.45) \quad E_z = E_z^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin \alpha} e^{ikz \cos \alpha} + \frac{n(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha} .$$

3. Region 4: $(2\pi - \theta_0 < \theta < \frac{3\pi}{2})$

$$(8.46) \quad E_x = E_x^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin \alpha} e^{ikz \cos \alpha} + E_x^{\text{ref}} e^{-ikr \cos(\theta + \theta_0) \sin \alpha} e^{ikz \cos \alpha} + \frac{l(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.47) \quad E_y = E_y^{inc} e^{-ikr \cos(\theta - \theta_0) \sin a} e^{ikz \cos a} + E_y^{ref} e^{-ikr \cos(\theta + \theta_0) \sin a} e^{ikz \cos a} \\ + \frac{m(\theta)}{\sqrt{r}} e^{ikr \sin a} e^{ikz \cos a}$$

$$(8.48) \quad E_z = E_z^{inc} e^{-ikr \cos(\theta - \theta_0) \sin a} e^{ikz \cos a} + E_z^{ref} e^{-ikr \cos(\theta + \theta_0) \sin a} e^{ikz \cos a} \\ + \frac{n(\theta)}{\sqrt{r}} e^{ikr \sin a} e^{ikz \cos a}$$

where

$$(8.49) \quad s\left(\frac{1}{s} \sin a \cos \theta_0 - 1\right) F_x^{ref} = E_x^{inc} \left\{ (\sin^2 a \sin^2 \theta_0 + \cos^2 a) s \right\} \\ - E_y^{inc} \left\{ \sin a \sin \theta_0 (1 + s \sin a \cos \theta_0) \right\} + E_z^{inc} \left\{ \cos a (1 + s \sin a \cos \theta_0) \right\}$$

$$(8.50) \quad -s D E_y^{ref} = E_x^{inc} \left\{ s \sin^2 a \sin \theta_0 \cos \theta_0 (1 - s \sin a \cos \theta_0) \right\} \\ + E_y^{inc} \left\{ (1 + s \sin a \cos \theta_0) \left[s (1 - \sin^2 a \sin^2 \theta_0) - \sin a \cos \theta_0 \right] \right\} \\ + E_z^{inc} \left\{ s \sin a \cos a \sin \theta_0 (1 + s \sin a \cos \theta_0) \right\}$$

$$(8.51) \quad -s D E_z^{ref} = E_x^{inc} \left\{ -s \cos a \sin a \cos \theta_0 (1 - s \sin a \cos \theta_0) \right\} \\ + E_y^{inc} \left\{ s \cos a \sin a \sin \theta_0 (1 + s \sin a \cos \theta_0) \right\} \\ + E_z^{inc} \left\{ s (1 + s \sin a \cos \theta_0) \sin a (\sin a - \cos \theta_0) \right\},$$

and D is given by (8.27).

Case C: $\pi < \theta_0 < \frac{3\pi}{2}$

1. Region 1: $(0 < \theta < \theta_0 - \pi)$

$$(8.52) \quad E_x = \frac{\ell(\theta)}{\sqrt{r}} e^{iksina} e^{ikzcosa}$$

$$(8.53) \quad E_y = \frac{m(\theta)}{\sqrt{r}} e^{ikrsina} e^{ikzcosa}$$

$$(8.54) \quad E_z = \frac{n(\theta)}{\sqrt{r}} e^{ikrsina} e^{ikzcosa}$$

2. Region 2: $(\theta_0 - \pi < \theta < 2\pi - \theta_0)$

$$(8.55) \quad E_x = E_x^{inc} e^{-ikrcos(\theta - \theta_0)sina} e^{ikzcosa} + \frac{\ell(\theta)}{\sqrt{r}} e^{ikrsina} e^{ikzcosa}$$

$$(8.56) \quad E_y = E_y^{inc} e^{-ikrcos(\theta - \theta_0)sina} e^{ikzcosa} + \frac{m(\theta)}{\sqrt{r}} e^{ikrsina} e^{ikzcosa}$$

$$(8.57) \quad E_z = E_z^{inc} e^{-ikrcos(\theta - \theta_0)sina} e^{ikzcosa} + \frac{n(\theta)}{\sqrt{r}} e^{ikrsina} e^{ikzcosa}$$

3. Region 3 and 4: $(2\pi - \theta_0 < \theta < \frac{3\pi}{2})$

$$(8.58) \quad E_x = E_x^{inc} e^{-ikrcos(\theta - \theta_0)sina} e^{ikzcosa} + E_x^{ref} e^{-ikrcos(\theta + \theta_0)sina} e^{ikzcosa} \\ + \frac{\ell(\theta)}{\sqrt{r}} e^{ikrsina} e^{ikzcosa}$$

$$(8.59) \quad E_y = E_y^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin \alpha} e^{ikz \cos \alpha} + E_y^{\text{ref}} e^{-ikr \cos(\theta + \theta_0) \sin \alpha} e^{ikz \cos \alpha} \\ + \frac{m(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

$$(8.60) \quad E_z = E_z^{\text{inc}} e^{-ikr \cos(\theta - \theta_0) \sin \alpha} e^{ikz \cos \alpha} + E_z^{\text{ref}} e^{-ikr \cos(\theta + \theta_0) \sin \alpha} e^{ikz \cos \alpha} \\ + \frac{n(\theta)}{\sqrt{r}} e^{ikr \sin \alpha} e^{ikz \cos \alpha}$$

where E_x^{ref} , E_y^{ref} and E_z^{ref} are given by (8.49), (8.50) and (8.51), respectively.

The expressions for $\ell(\theta)$, $m(\theta)$ and $n(\theta)$ are given by (8.24'), (8.25') and (8.26') respectively. Note that E_x^{inc} , E_y^{inc} and E_z^{inc} are not all independent but are related by the divergence condition given by (4.22).

Appendix

Discussion of the impedance boundary condition

On the surface $x = 0, y \leq 0$ of the right-angled wedge, the following boundary condition is imposed:

$$(A.1) \quad \begin{pmatrix} E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 0 & Z \\ -Z & 0 \end{pmatrix} \begin{pmatrix} H_y \\ H_z \end{pmatrix} \quad x = 0, y \leq 0.$$

In this appendix we wish to show that if

$$(A.2) \quad Z = \frac{\omega \mu}{k} s$$

where s is real and positive, then these conditions represent the behavior at the surface of a partial absorber. By a partial absorber we mean a surface whose absorption depends upon the direction of the propagation vector of the incident plane wave and the material of the surface.

Since we are interested, at the moment, in the properties of a surface characterized by a given impedance boundary condition, and not in the actual solution of the wedge problem, we shall treat the simplified problem of the reflection of an incident plane wave by a plane surface of infinite extent. The impedance boundary condition we employ is

$$(A.3) \quad \begin{pmatrix} E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 0 & Z \\ -Z & 0 \end{pmatrix} \begin{pmatrix} H_y \\ H_z \end{pmatrix} \quad x = 0.$$

We note that since we have an infinite plane boundary, there is no preferred orientation for the direction of the incident wave. Hence, without loss of generality, we can assume that the propagation vector of the incoming plane

wave lies in the x-y plane and that this plane is the plane of incidence. See Figure A.1. The problem is now essentially two-dimensional and we need to consider only the two cases in which the electric vector is either perpendicular or parallel to the plane of incidence. In the first case, the magnetic vector is parallel to the plane of incidence and hence $H_z \equiv 0$. In the second case, the magnetic vector is perpendicular to the plane of incidence and hence $H_x = H_y \equiv 0$. The physical significance of the boundary conditions used in the wedge problem can be determined from a study of this simplified problem.

Suppose we first consider the case in which the electric vector is perpendicular to the plane of incidence. Then

$$(A.4) \quad E_z = E_z^{\text{inc}} + E_z^{\text{ref}}$$

where

$$(A.5) \quad E_z^{\text{inc}} = e^{ik(-x \cos \theta_o - y \sin \theta_o)}$$

and

$$(A.6) \quad E_z^{\text{ref}} = R e^{ik(+x \cos \theta_o - y \sin \theta_o)}$$

The boundary condition appropriate for this polarization is

$$(A.7) \quad E_z = -Z H_y \quad x = 0$$

or

$$(A.8) \quad \frac{\partial E_z}{\partial x} = \frac{ik}{s} E_z \quad x = 0$$

where we have used one of Maxwell's equations and the fact that $E_x = E_y = 0$.

The reflection coefficient $R_{\perp}$ is easily shown to be

$$(A.9) \quad R_{\perp} = \frac{s \cos \theta_o + 1}{s \cos \theta_o - 1} \quad .$$

From Figure A.1 we see that $\pi/2 < \theta_0 < 3\pi/2$. Hence $\cos \theta_0$ is always negative and $R_{\perp}$ is finite for all admissible values of θ_0 provided s is real and positive. The reflection coefficient is plotted as a function of the angle of incidence θ_0 and the positive real number s in Figure A.2. We note especially that if $s = 1$, the reflection coefficient $R_{\perp}$ has a minimum value of zero when the incident wave is perpendicular to the surface. It increases for oblique incidence and is a maximum (unity) at grazing incidence. The zero value of the reflection coefficient corresponds to complete absorption. When $s < 1$ the reflection coefficient $R_{\perp}$ has a minimum (non-zero) when the incident wave is perpendicular to the surface and is a maximum (unity) at grazing incidence. When $s > 1$ the same general behavior exists except that we have a zero reflection coefficient for angles of incidence other than normal incidence. We see that for suitably chosen s we can obtain complete absorption for any arbitrary angle of incidence. Thus the reflection coefficient is a function of the angle of incidence and the dimensionless real positive number s , and behaves in a reasonable physical manner. When $s = 0$ we can speak of the surface as being a perfect conductor or a perfect reflector. For $s \neq 0$, the surface might be called an imperfect reflector or, as we prefer, a partial absorber. Therefore, we say that the surface has the behavior of a partial absorber.

Let us next consider the case in which the electric vector is parallel to the plane of incidence. Then

$$(A.10) \quad H_z = H_z^{\text{inc}} + H_z^{\text{ref}}$$

where

$$(A.11) \quad H_z^{\text{inc}} = e^{ik(-x\cos\theta_0 - y\sin\theta_0)}$$

and

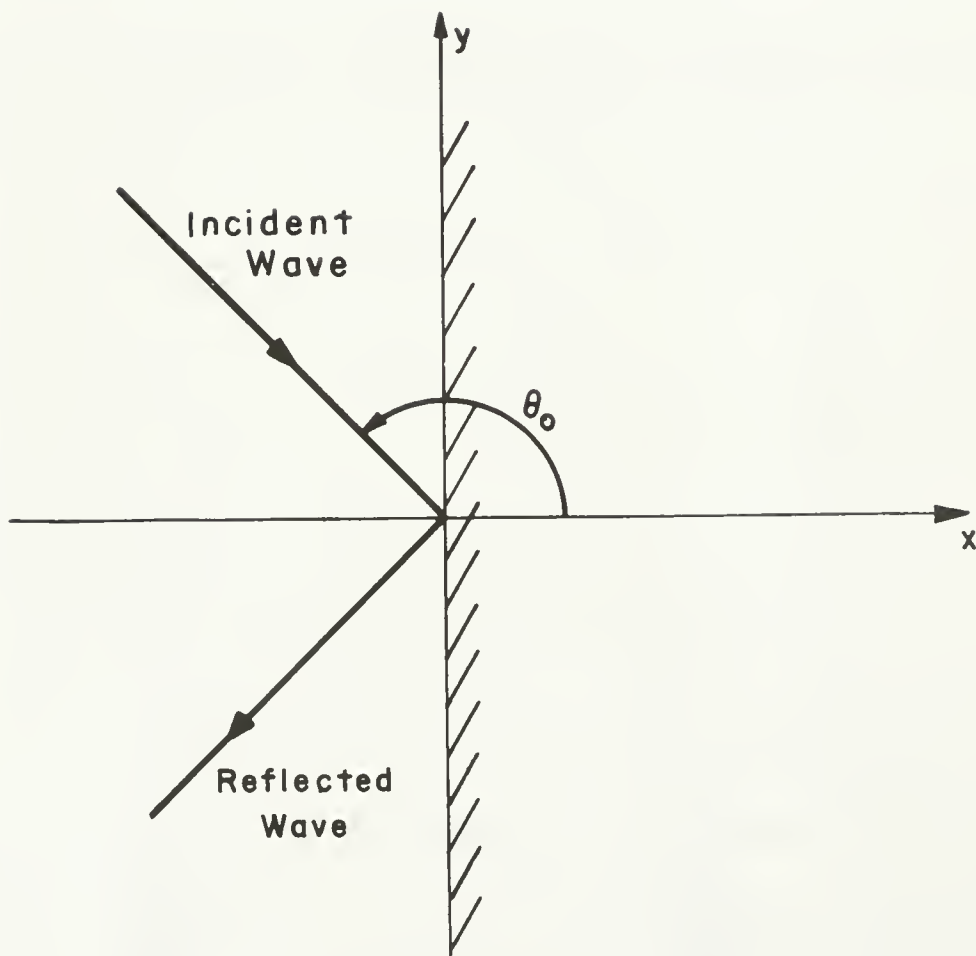


Figure A-1

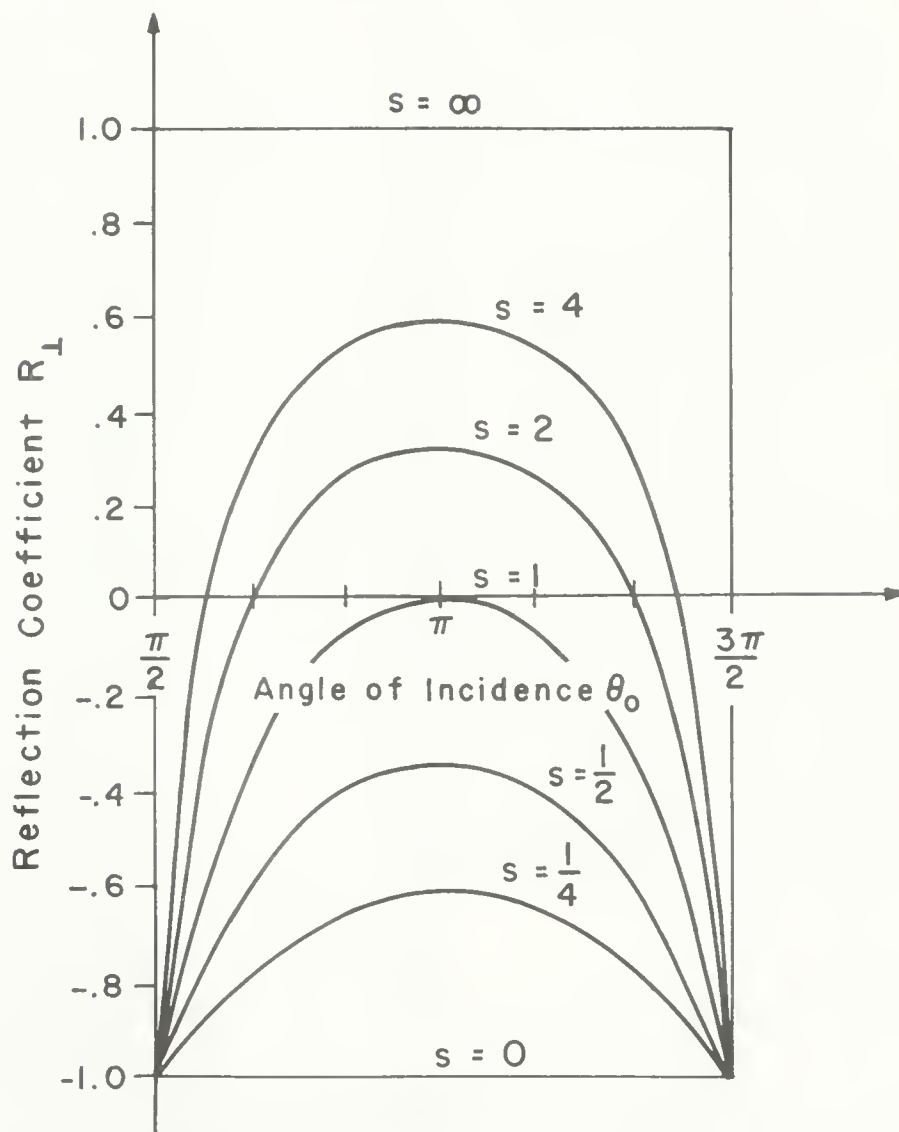


Figure A-2

$$(A.12) \quad H_z^{\text{ref}} = R_{\parallel} e^{ik(+x \cos \theta_o - y \sin \theta_o)}.$$

The boundary condition appropriate for this polarization is

$$(A.13) \quad E_y = Z H_z, \quad x = 0$$

or

$$(A.14) \quad \frac{\partial H_z}{\partial x} = -\frac{k^2}{\lambda} H_z, \quad x = 0$$

where we have used one of Maxwell's equations and the fact that $H_x = H_y = 0$.

The reflection coefficient $R_{\parallel}$ is easily shown to be

$$(A.15) \quad R_{\parallel} = \frac{\frac{1}{s} \cos \theta_o + 1}{\frac{1}{s} \cos \theta_o - 1}.$$

The reflection coefficient $R_{\parallel}$ is plotted in Figure A.3. This reflection coefficient is also reasonable on physical grounds and characterizes the behavior of an absorber.

Let us apply these results to our wedge problem. In our problem the absorbing surface is chosen as the y - z plane. Our restricted form of the impedance condition then allows, in view of the results of the preceding paragraphs, the representation of complete absorption for a given angle of incidence provided either the incident electric or magnetic field has a vanishing z component. The latter requirement can be fulfilled if and only if the incident propagation vector lies in the x , y plane, i.e., in the plane perpendicular to the edge of the wedge. Then in our notation $\alpha = \pi/2$. In this case, given any angle of incidence θ_o , where $\pi/2 < \theta_o < 3\pi/2$, we can assign a value of s for perfect absorption as follows:

$$(A.16) \quad \begin{aligned} E_z^{\text{inc}} &= 0, & s &= -\cos \theta_o \\ H_z^{\text{inc}} &= 0, & s &= -\sec \theta_o. \end{aligned}$$

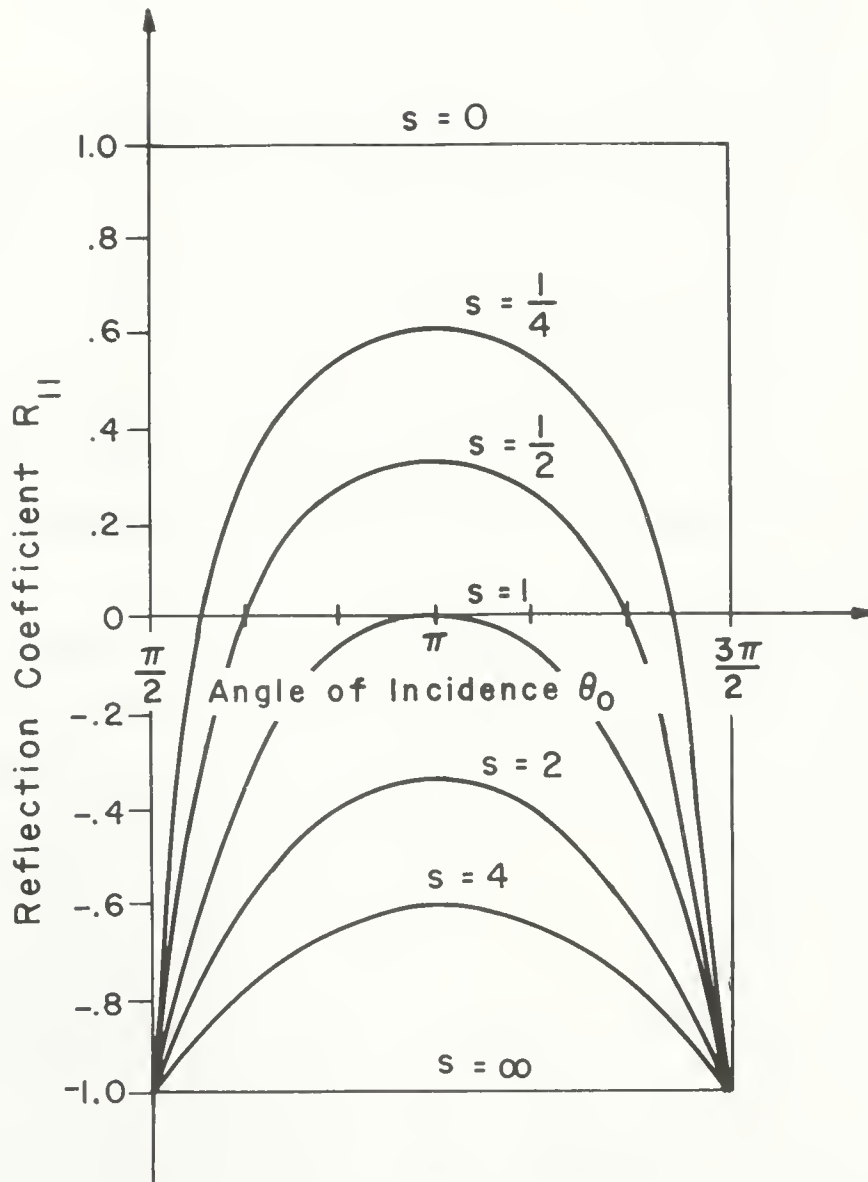


Figure A-3

In addition, our restricted phenomenological model of the structure of the absorbing screen, as expressed in our form of impedance matrix, allows the representation of complete absorption for one special value of θ_0 , namely $\theta_0 = \pi$ (normal incidence) even when F_z^{inc} and H_z^{inc} are both present in finite amounts. A glance at Figures A.2 and A.3 shows that in this case $s = 1$ is necessary and sufficient for complete absorption.

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